

Mathematical analysis of a coarsening model with local interactions

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Abstract

We consider particles on a one-dimensional lattice whose evolution is governed by nearest-neighbor interactions where particles that have reached size zero are removed from the system. Concentrating on configurations with infinitely many particles, we prove existence of solutions under a reasonable density assumption on the initial data and show that the vanishing of particles and the localized interactions can lead to non-uniqueness. Moreover, we provide a rigorous upper coarsening estimate and discuss generic statistical properties as well as some non-generic behavior of the evolution by means of heuristic arguments and numerical observations.

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1 Introduction

Discrete systems with local interactions are used in a wide range of models for growth processes. Examples are the coarsening of sand ripples [HK02] or the clustering in granular gases [vdMvdWL01], where matter is transported between adjacent ripples or clusters, and certain particle hopping models [Spi70], where particles occupy cells on a one-dimensional lattice and jump between neighboring cells with rates depending on the particle density.

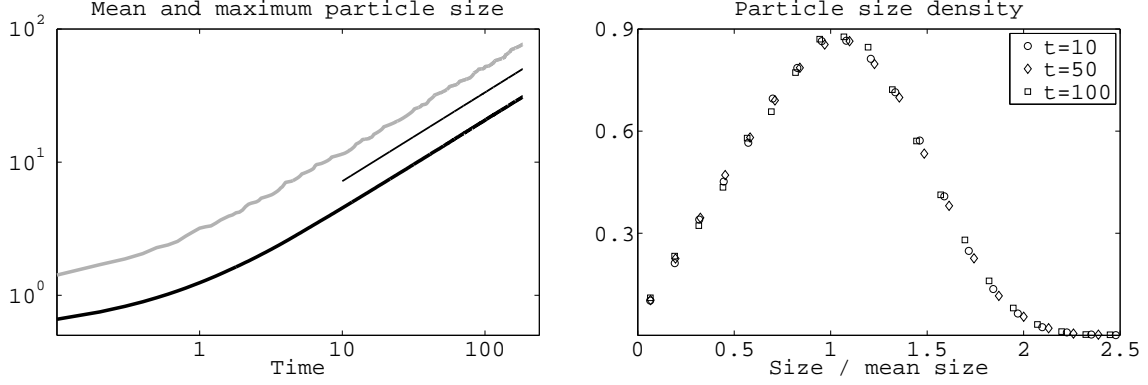


Figure 1.1: Numerical simulation of (1.1) with $\beta = 0.5$ and initial particle sizes uniformly distributed in the interval $(0, 1)$. *Left.* Mean particle size (black) and maximum particle size (gray) over time, both in logarithmic scale. The thin line represents the power law $t \mapsto t^{1/(\beta+1)}$. *Right.* Particle size density scaled by the mean size at three different times during the evolution.

Discrete systems with local interactions also appear as reduced models of more complex situations such as the coarsening of droplets in dewetting thin films [GW03, GW05], the evolution of phase domains in the convective Cahn-Hilliard equation [WORD03], or the motion of grain boundaries in polycrystalline materials [HNO04]. Furthermore, local interactions emerge in discrete or discretized ill-posed diffusion equations such as the Perona-Malik equation used in image segmentation [PM90] or population dynamics [HPO04], and there they lead to localized aggregation and coarsening as well [EG09, ES08].

In this paper, we consider a discrete system of particles of sizes (x_j) , $j \in \mathbb{Z}$ whose evolution for time $t > 0$ is governed by

$$\dot{x}_j = x_{j+1}^{-\beta} - 2x_j^{-\beta} + x_{j-1}^{-\beta} \quad (1.1)$$

as long as $x_j > 0$; particles that have reached $x = 0$ are removed from the system and the remaining ones are relabeled. The parameter β is a positive real number. Equation (1.1) is closely related to the first examples above where our particles represent sand ripples, gas clusters or lattice cells. It can also be regarded as the discretized backward-parabolic equation $x_j = \Delta g(x_j)$ with decreasing mass transfer function $g(s) = s^{-\beta}$ and Δ denoting the discrete Laplacian on \mathbb{Z} . It differs from the discrete Perona-Malik equation, though, since the latter has a forward-parabolic region near the origin, whereby small particles do not vanish but cluster near the boundary to the backward-parabolic region. Moreover, due to its rather simple structure we see (1.1) as a useful toy problem for studying local correlations in coarsening systems and thus as a step towards understanding more complex and higher dimensional evolutions such as grain growth or the Mullins-Sekerka flow with positive volume fraction.

The most interesting aspect of (1.1) and the examples above are the universal or generic statistical properties, which the evolutions seem to exhibit. Often, these can be derived formally from the equation or observed in numerical simulations of large finite systems, and for solutions to (1.1) the key findings are the following.

1. Particles disappear in finite time, whereas the total particle size $\sum_{j \in \mathbb{Z}} x_j$ is formally conserved during the evolution. Thus, the typical particle size grows, and a dimensional analysis suggests the growth law

$$\text{typical particle size} \sim t^{\frac{1}{\beta+1}}. \quad (1.2)$$

The latter is confirmed by numerical results as depicted in Figure 1.1.

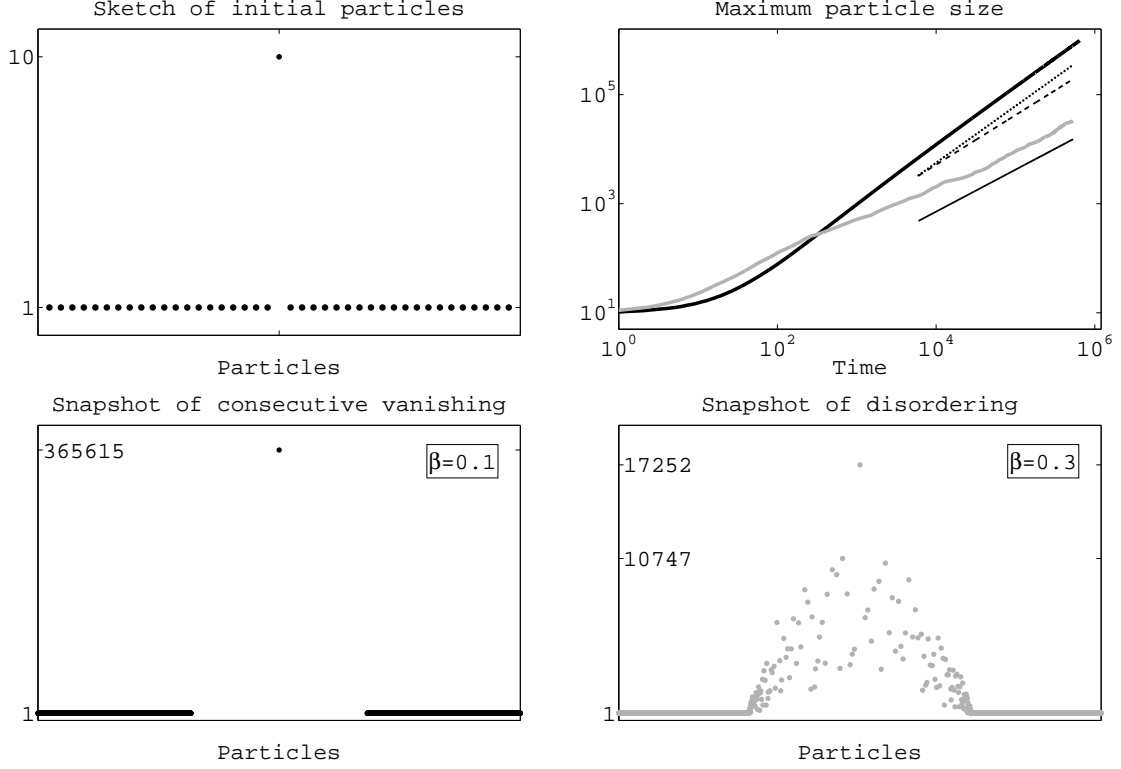


Figure 1.2: *Top left.* Sketch of initial data where one large particle, here $x_0 = 10$, is surrounded by small particles $x_j = 1$. *Bottom row.* Snapshots of the evolutions for $\beta = 0.1$ (left, black) and $\beta = 0.3$ (right, gray) at a later time. The small black particles vanish sequentially beginning at the large one, whereas the small gray particles become disordered. *Top right.* Maximum particle size for both systems over time, in logarithmic scale. For $\beta = 0.1$ the largest particle grows linearly in time (represented by the thin dotted line) instead of adhering to the generic growth law (1.2) (thin dashed line), while for $\beta = 0.3$ its growth approaches the generic law (thin solid line). See Section 2.3 for a more detailed discussion of the underlying dynamics.

2. The simulations in Figure 1.1 also indicate self-similarity of the particle size distribution in the sense

$$\text{particle size density}(t, x) \sim t^{-\frac{1}{\beta+1}} p\left(xt^{-\frac{1}{\beta+1}}\right)$$

for some profile $p: [0, \infty) \rightarrow [0, \infty)$.

3. Not all solutions display the behavior just described. For instance, constant initial data remain constant for all times, and as indicated in Figure 1.2 there are solutions where certain particles grow faster than the law (1.2).

The analysis of coarsening in equations as (1.1) therefore comprises two problems: first, to derive an appropriate upper bound for the (typical) particle size that is true for all solutions, and second, to find conditions on the initial data that guarantee the generic behavior of corresponding solutions. The second issue is in general very hard to deal with and any mathematical consideration must be tailored to the specific evolution, whereas upper coarsening estimates have been obtained for several particle growth processes using numerical simulations, heuristic arguments or variants of the method introduced in [KO02]. Usually, however, it is difficult to apply these approaches rigorously to infinite particle systems.

In the dynamics of (1.1), on the other hand, finitely many particles generically disappear in finite time, and due to vanishings and local interactions there is also no evident equation

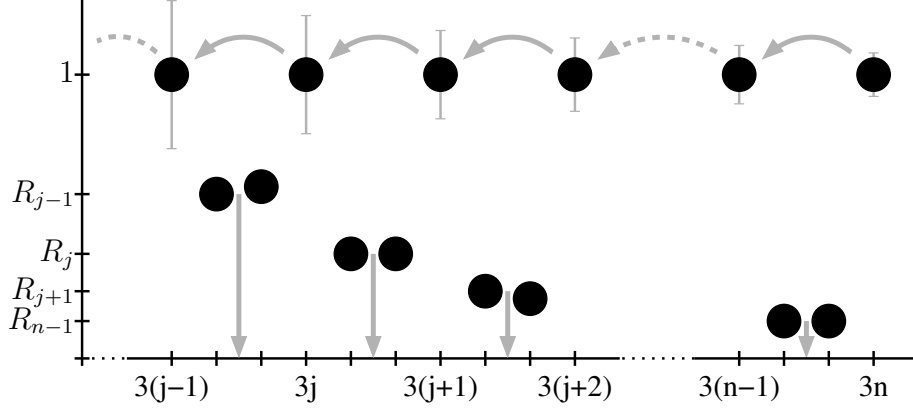


Figure 1.3: Sketch of the initial data for our non-uniqueness result. Large particles $x_{3j} \sim 1$, $j > 0$ are followed by small ones $R_{3j+1} \sim R_{3j+2} \sim R_j$ where $R_j \rightarrow 0$ as $j \rightarrow \infty$. The small particles vanish pairwise from right to left, and thereby tiny perturbations of a large particle (thin gray error bars) are transported to the left and amplified. Non-uniqueness is then a consequence of the infinite number of particles, which allows arbitrarily large distances to be covered and thus amplifications of order 1.

for the evolution of a particle size distribution. In order to rigorously characterize the statistical properties, our goal is thus the mathematical analysis of infinite systems. In the current paper, we study well-posedness of (1.1) for infinitely many particles and prove existence of solutions under a density condition on the initial distribution. More precisely, assuming that there are two constants $L > 0$ and $d > 0$ such that

$$\frac{1}{L} \sum_{j \in R} x_j(0) \geq d$$

for any region $R \subset \mathbb{Z}$ containing L consecutive particles we show that there are locally Hölder continuous trajectories $x_j: [0, \infty) \rightarrow [0, \infty)$ that satisfy (1.1) in an integral sense and almost everywhere as long as $x_j > 0$. In particular, a solution contains infinitely many positive particles at any time $t \geq 0$.

It has been observed that infinite particle systems might have pathological solutions, for instance in [Lan75] where the author discusses a Hamiltonian system of classical particles interacting by means of hard core potentials. He demonstrates that for well-prepared initial data it is possible to transfer energy from infinity to a finite region in space via infinitely many particle collisions in finite time, and along a similar idea we show that solutions to (1.1) are not uniquely determined by their initial data. To this end, we consider a configuration as sketched in Figure 1.3 where large particles $x_{3j} \sim 1$ are followed by two small ones $x_{3j+1} \sim x_{3j+2} \sim R_j$ with $R_j \rightarrow 0$ as $j \rightarrow \infty$. Exploiting the effects of vanishings and the localized interactions we prove that (even in finite systems) tiny perturbations pass through the configuration and are amplified during the evolution. Arbitrarily small initial perturbations may reach a size of order 1 if the distance they travel, and thus their total amplification, is sufficiently large. In the infinite system, we then obtain two different solutions by considering unperturbed data as well as decreasing perturbations at increasing particle indices.

As a first step towards understanding the statistical properties of (1.1), we provide a simple upper coarsening estimate relating the mean particle size of a system and the time that is necessary to transport a fraction of it from vanishing particles to remaining ones. We also discuss part of the non-generic behavior indicated in Figure 1.2 relying on heuristic arguments for the mass transfer in the system and numerical investigations.

The rest of the paper is organized as follows. In Section 2 we introduce our precise setting, which in particular comprises how we deal with vanishing particles. We also collect our coarsening results in Sections 2.2 and 2.3. The existence theorem and its proof are contained in Section 3, while Section 4 is devoted to non-uniqueness.

2 Setup and coarsening

A precise description of the dynamics of (1.1) requires that we keep track of particles and their neighborhoods throughout the evolution. For this reason we consider sequences $x = (x_j)$ in the state space

$$\ell_+^\infty = \{x = (x_j) : 0 \leq x_j \leq C \text{ for all } j \in \mathbb{Z} \text{ and some constant } C > 0\},$$

which include *vanished* particles $x_j = 0$ as well as *living* ones $x_j > 0$; the term *particle* can refer to the size x_j as well as the index j . Since only adjacent living particles interact with each other, we use the term *neighborhood* of a particle x_j to denote the two nearest living particles $x_{\sigma_\pm(j,x)}$ where

$$\sigma_-(j, x) = \sup \{k < j : x_k > 0\}, \quad \sigma_+(j, x) = \inf \{k > j : x_k > 0\}; \quad (2.1)$$

in the case that one of the sets in (2.1) is empty we set $\sup \emptyset = -\infty$, $\inf \emptyset = \infty$ and $x_{\pm\infty} = 0$. We also write

$$\sigma_-(\infty, x) = \sup \{k : x_k > 0\}, \quad \sigma_+(-\infty, x) = \inf \{k : x_k > 0\}$$

for the largest and smallest particle index in a finite system, respectively.

Given a function $g: [0, \infty) \rightarrow [0, \infty)$, we denote by

$$\Delta_\sigma g(x_j) = \Delta_{\sigma(j,x)} g(x_j) = \left(g(x_{\sigma_-(j,x)}) - 2g(x_j) + g(x_{\sigma_+(j,x)}) \right) \cdot \chi_{\{x_j > 0\}}$$

the *living-particles-Laplacian* of the sequence $g(x)$; for simplicity of notation we omit the indices j and x in σ , σ_\pm when the meaning is clear from the context. In our model problem we have

$$g(s) = s^{-\beta} \quad \text{for } 0 < s < \infty, \quad g(0) = 0,$$

where $\beta > 0$, and with the convention $0^{-\beta} = 0$ we write (1.1) as

$$\dot{x}_j(t) = \Delta_\sigma g(x_j(t)) = \Delta_\sigma x_j(t)^{-\beta} \quad \text{for } j \in \mathbb{Z}, t > 0 \quad (2.2)$$

or

$$\dot{x}(t) = \Delta_\sigma x(t)^{-\beta} \quad \text{for } t > 0.$$

By definition of Δ_σ , the right hand side of (2.2) is well-defined for all $j \in \mathbb{Z}$. In particular, we have $\dot{x}_j(t) = 0$ if $x_j(t) = 0$ and $\dot{x}_j(t) = x_{j-1}(t)^{-\beta} - 2x_j(t)^{-\beta}$ if $x_{j-1}(t) > 0$, $x_j(t) > 0$, but $\sigma_+(j, x(t)) = \infty$. Finally, if $x_j(t) > 0$ for all $j \in \mathbb{Z}$, then Δ_σ is the standard discrete Laplace operator on \mathbb{Z} .

2.1 Notion of solution

We now define our notion of solution. As we will prove existence by passing to the limit in the number of particles in a system, an integral formulation is appropriate. On the other hand, for a priori estimates the differential equation (2.2) is more convenient and hence we briefly discuss equivalence of both formulations.

Definition 2.1 (Integral solution). For $T \in (0, \infty]$ we call $x = (x_j)_{j \in \mathbb{Z}} : [0, T) \rightarrow \ell_+^\infty$ a solution to (2.2) with initial data $x(0)$ if

1. each x_j is continuous in $[0, T)$;
2. each function $x_j^{-\beta} \chi_{\{x_j > 0\}}$, where $0^{-\beta} \cdot 0 = 0$, is locally integrable in $[0, T)$;
3. for any $j \in \mathbb{Z}$ the equation

$$x_j(t_2) - x_j(t_1) = \int_{t_1}^{t_2} \Delta_\sigma x_j(s)^{-\beta} ds \quad (2.3)$$

is satisfied for all $0 \leq t_1 < t_2 < T$.

For a solution x as in Definition 2.1 and $m, n \in \mathbb{Z}$, $m \leq n$, we denote by

$$M_{m,n}(t) = \sum_{k=m}^n x_k(t) \quad (2.4)$$

the mass of the particles m, \dots, n at time $t \in [0, T)$. Using (2.3) and the definition of Δ_σ , we compute

$$M_{m,n}(t_2) - M_{m,n}(t_1) = \int_{t_1}^{t_2} \left(x_{\sigma_-(m)}^{-\beta} - x_{\sigma_+(\sigma_-(m))}^{-\beta} - x_{\sigma_-(\sigma_+(n))}^{-\beta} + x_{\sigma_+(n)}^{-\beta} \right) \cdot \chi_{\{M > 0\}} ds \quad (2.5)$$

for $0 \leq t_1 < t_2 < T$, where $M_{m,n}(t) > 0$ implies that there is at least one living particle among m, \dots, n and thus that $\sigma_-(m) < m \leq \sigma_+(\sigma_-(m)) \leq \sigma_-(\sigma_+(n)) \leq n < \sigma_+(n)$ in the integrand on the right hand side.

The models in the introduction assume that vanished particles do not reappear at later times, and with the help of (2.5) we can show that our solutions indeed have this property.

Lemma 2.2. *If $x_j(t_1) = 0$ for some $j \in \mathbb{Z}$ and $t_1 \in [0, T)$, then $x_j(t) = 0$ for all $t \in [t_1, T)$.*

Proof. Assume for contradiction that the claim is wrong and denote by

$$t_2 = \inf \{t > t_1 : x_j(t) > 0\} \in [t_1, T)$$

the time up to which x_j remains vanished. Suppose first that $m = \sigma_-(j, x(t_2))$ and $n = \sigma_+(j, x(t_2))$ are both finite and fix $t_3 \in (t_2, T)$ so that by continuity of particle sizes we have

$$x_m(t)^{-\beta} \leq c, \quad x_n(t)^{-\beta} \leq c \quad \text{and} \quad x_k(t)^{-\beta} \geq 2c, \quad k = m+1, \dots, n-1$$

for all $t \in [t_2, t_3]$ and some constant $c > 0$. The change of $M_{m+1,n-1}(t)$ then satisfies

$$x_m(t)^{-\beta} - x_{\sigma_+(m, x(t))}(t)^{-\beta} - x_{\sigma_-(n, x(t))}(t)^{-\beta} + x_n(t)^{-\beta} \leq -2c, \quad (2.6)$$

for $t \in [t_2, t_3]$ such that $M_{m+1,n-1}(t) > 0$, hence (2.5) gives

$$M_{m+1,n-1}(t) \leq -2c |\{s \in (t_2, t) : M_{m+1,n-1}(s) > 0\}|$$

for all $t \in [t_2, t_3]$. This is a contradiction to $0 < x_j(t) \leq M_{m+1,n-1}(t)$ for t sufficiently close to t_2 .

In the cases that $m = -\infty$ or $n = \infty$ or both, the corresponding positive term(s) in (2.6) disappear(s). Since, however, the negative contribution $-x_{\sigma_+(m)}(t)^{-\beta} - x_{\sigma_-(n)}(t)^{-\beta}$ remains, we still obtain the contradiction as above. \square

Remark (Vanishing times). As a consequence of Lemma 2.2 each particle x_j has a unique *vanishing time* $\tau_j \in [0, T) \cup \{\infty\}$ such that

$$x_j(t) > 0 \text{ for } t < \tau_j \quad \text{and} \quad x_j(t) = 0 \text{ for } t \geq \tau_j,$$

where we set $\tau_j = \infty$ if and only if $x_j(t) > 0$ for all $t \in [0, T)$. Moreover, the functions $t \mapsto \sigma_{\pm}(j, x(t))$ are monotone, right-continuous and change their value in vanishing times only.

Lemma 2.3 (Solving the ODE). *For $j \in \mathbb{Z}$ let $\mathcal{V}_j = \{t : \sigma_{\pm}(j, x(t)) \text{ is not continuous}\}$ be the set of times when a neighbor of x_j vanishes. Then x_j is continuously differentiable for all $t \in (0, T) \setminus (\{\tau_j\} \cup \mathcal{V}_j)$ and satisfies (2.2).*

Proof. Fix $j \in \mathbb{Z}$. For $t > \tau_j$ the claim is trivial and for $t \in (0, \tau_j) \setminus \mathcal{V}_j$ it follows from the Fundamental Theorem of Calculus applied to (2.3) in a sufficiently small time interval around t . More precisely, with $m = \sigma_{-}(j, x(t))$ and $n = \sigma_{+}(j, x(t))$ fixed, continuity of x_j , x_m and x_n provides $\delta > 0$ and $c > 0$ such that

$$\min\{x_m(s), x_j(s), x_n(s) : s \in (t - \delta, t + \delta)\} \geq c > 0.$$

Hence, $\sigma_{-}(j, x(s)) \geq m$ and $\sigma_{+}(j, x(s)) \leq n$ for all $s \in (t - \delta, t + \delta)$, which implies that as functions of s both change their value only at finitely many times in $\mathcal{V}_j \cap (t - \delta, t + \delta)$. As t is not such a time, we may decrease δ to obtain $\sigma_{-}(j, x(s)) = m$ and $\sigma_{+}(j, x(s)) = n$ for all $s \in (t - \delta, t + \delta)$. Therefore,

$$\frac{x_j(t+h) - x_j(t)}{h} = \frac{1}{h} \int_t^{t+h} x_m(s)^{-\beta} - 2x_j(s)^{-\beta} + x_n(s)^{-\beta} \, ds$$

for all $|h| < \delta$, and the claim follows as stated above. \square

Remark. The preceding lemmas show that every solution x has the property that

1. for each j there exists a unique vanishing time τ_j ;
2. each x_j is continuously differentiable in $(0, T) \setminus \mathcal{V}$, where $\mathcal{V} = \{\tau_k : k \in \mathbb{Z}\}$, and equation (2.2) holds.

The other way round, if $x : [0, T) \rightarrow \ell_+^{\infty}$ satisfies the second item above with \mathcal{V} replaced by an arbitrary null set of $[0, T)$ and Definition 2.1(1 and 2), then also the integral identity (2.3) is true. Moreover, uniqueness of vanishing times follows directly from the differential equation if the latter holds almost everywhere, since the argument in the proof of Lemma 2.2 is easily adapted.

2.2 Upper coarsening estimate

Equation (2.2) can formally be interpreted as the H^{-1} gradient flow on living particles of the energy

$$E(x) = \frac{1}{1-\beta} \sum_{k: x_k > 0} x_k^{1-\beta} \quad \text{if } \beta \neq 1 \quad \text{or} \quad E(x) = \sum_{k: x_k > 0} \ln x_k \quad \text{if } \beta = 1,$$

and such a structure has proved successful for obtaining upper coarsening bounds in similar (finite) systems; see for instance [EG09, ES08] for the application of [KO02] to a discrete H^{-1} gradient flow. However, besides the fact that energy and metric tensor of infinite systems are in general infinite, this gradient flow structure does not seem useful here, since

(even in finite systems) at each vanishing time the local correlations in the metric tensor change and for $\beta \geq 1$ the energy tends to negative infinity. We thus take a different approach, which regards an upper coarsening estimate as a lower bound on the time needed to transport a certain amount of mass through the system and which is derived from the basic inequality $\dot{x}_j \geq -2x_j^{-\beta}$.

For illustration, suppose that $x = (x_k)_{k=-n, \dots, n}$ is a solution to (2.2) with $2n+1$ nonzero initial particles whose mass is $M = \sum_{|k| \leq n} x_k(0)$ and denote by

$$T = \inf \left\{ t > 0 : \sum_{x_k(t)=0} x_k(0) \geq \frac{1}{2}M \right\}$$

the smallest time such that half of the initial mass has been transported from vanished particles to living ones. Then, there must exist a particle x_j with vanishing time $\tau_j \leq T$ and initial size at least $M/(2(2n+1))$, because otherwise we would have $\sum_{x_k(T)=0} x_k(0) < M/2$. From $\dot{x}_j \geq -2x_j^{-\beta}$ we infer that $x_j(0)^{\beta+1} \leq 2(\beta+1)\tau_j$ and hence obtain the estimate

$$\frac{1}{2(2n+1)}M \leq x_j(0) \leq (2(\beta+1)\tau_j)^{\frac{1}{\beta+1}} \leq [2(\beta+1)]^{\frac{1}{\beta+1}} T^{\frac{1}{\beta+1}},$$

which bounds $T^{1/(\beta+1)}$ from below by the initial mean particle size and a constant depending on β and the fraction of mass that is transported.

Assuming that the mean particle size initially exists, we prove a similar estimate for infinite particle systems.

Proposition 2.4. *Let $x: [0, \infty) \rightarrow \ell_+^\infty$ be a solution to (2.2) which initially has finite mean particle size*

$$X = \lim_{n \rightarrow \infty} X_n \in (0, \infty), \quad X_n = \frac{1}{2n+1} \sum_{j=-n}^n x_j(0)$$

and let

$$T_n = \inf \left\{ t > 0 : \frac{1}{2n+1} \sum_{\substack{j=-n, \dots, n \\ x_j(t)=0}} x_j(0) \geq \frac{1}{2}X_n \right\}$$

as well as

$$T = \inf \left\{ t > 0 : \liminf_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{\substack{j=-n, \dots, n \\ x_j(t)=0}} x_j(0) \geq \frac{1}{2}X \right\}.$$

Then there is a constant $c > 0$ such that $\liminf_{n \rightarrow \infty} T_n \geq cX^{\beta+1}$ and $T \geq cX^{\beta+1}$.

In Proposition 2.4 both, T and $\liminf T_n$ are a measure for the time needed to transport half of the initial mean particle size from vanishing particles to living ones, but we do not know which one is better or more useful. Furthermore, the result only applies when sufficiently many particles do indeed vanish. We expect that this is the generic case for long-time evolutions, since the discrete backward-parabolic equation (2.2) increases differences among particle sizes and since small particles surrounded by larger ones disappear sufficiently fast; see Lemma 3.6. On the other hand, there are situations such that no or very few particles vanish and the possible rate of coarsening is not captured by our estimate; consider for instance a system of large regions of equal particles and compare the mass concentration example in Section 2.3.

Proof of Proposition 2.4. As for finite systems above we obtain $T_n \geq cX_n^{\beta+1}$, and taking the lower limit as $n \rightarrow \infty$ proves the first inequality.

By definition of T there is for any $\varepsilon > 0$ a time $t_\varepsilon \in [T, T + \varepsilon]$ such that

$$\liminf_{n \rightarrow \infty} V_n(t_\varepsilon) \geq \frac{1}{2}X, \quad V_n(t) = \frac{1}{2n+1} \sum_{\substack{j=-n, \dots, n \\ x_j(t)=0}} x_j(0).$$

Hence, there is $n_0 = n_0(\varepsilon)$ such that $V_n(t_\varepsilon) \geq X/2 - \varepsilon$ for all $n > n_0$ and the argument for finite systems yields

$$\frac{1}{2}X - \varepsilon \leq ct_\varepsilon^{\frac{1}{\beta+1}} \leq c(T + \varepsilon)^{\frac{1}{\beta+1}}.$$

Sending $\varepsilon \rightarrow 0$ finishes the proof. \square

2.3 Example: mass concentration versus mass spreading

We conclude this section with a heuristic discussion of the example shown in Figure 1.2 where initially a large $x_0(0) \gg 1$ is surrounded by $x_j(0) = 1$, $j \neq 0$.

The particle x_0 clearly grows during the evolution, and assuming that it is sufficiently large we may neglect its contribution to the particle velocities. Furthermore, due to the symmetry $x_{-j} = x_j$ for $j \in \mathbb{Z}$ it suffices to consider only the particles x_j , $j \geq 0$, and the dynamics of the example are effectively governed by

$$\dot{x}_0 \sim 2x_1^{-\beta}, \quad \dot{x}_1 \sim -2x_1^{-\beta} + x_2^{-\beta}, \quad \dot{x}_j \sim x_{j-1}^{-\beta} - 2x_j^{-\beta} + x_{j+1}^{-\beta} \quad \text{for } j > 1. \quad (2.7)$$

We let

$$x_j(t) = 1 + \frac{u_j(s)}{\beta}, \quad s = \beta t$$

and linearize

$$x_j^{-\beta}(t) = \exp(-\beta \ln x_j(t)) \sim e^{-u_j(s)},$$

so that (2.7) becomes

$$\frac{d}{ds}u_0 \sim 2e^{-u_1}, \quad \frac{d}{ds}u_1 \sim -2e^{-u_1} + e^{-u_2}, \quad \frac{d}{ds}u_j \sim e^{-u_{j-1}} - 2e^{-u_j} + e^{-u_{j+1}} \quad (2.8)$$

with initial data $u_0(0) \gg 1$ and $u_j(0) = 0$, $j > 0$. The initial conditions immediately show that

$$\frac{d}{ds}u_0(0) \sim 2 > 0, \quad \frac{d}{ds}u_1(0) \sim -1 < 0, \quad \frac{d}{ds}u_j(0) \sim 0 \quad \text{for } j > 1,$$

and by induction it is easy to see that

$$(-1)^j \left(\frac{d}{ds}\right)^j u_j(0) \sim 1 > 0, \quad \left(\frac{d}{ds}\right)^k u_j(0) = 0 \quad \text{for } k > j > 1.$$

Indeed, for $j > 0$ equation (2.8) reads $\frac{du_j}{ds} \sim \Delta e^{-u_j}$ where Δ is the discrete Laplacian on \mathbb{Z} , and taking the k -th time derivative we obtain

$$\left(\frac{d}{ds}\right)^k u_j \sim \Delta \left(\frac{d}{ds}\right)^{k-1} e^{-u_j},$$

which enables the induction argument. As a consequence, we find $(-1)^j u_j(s) > 0$ for small positive times s , and using these inequalities in (2.8), we conclude that they remain true for all s . Moreover, a comparison argument for $u_{j+2} - u_j$ shows that $u_0 \geq u_2 \geq u_4 \dots$ and $\dots \geq u_5 \geq u_3 \geq u_1$. In terms of the particles x_j this means that those with an even index j grow while those with an odd index shrink as long as (2.8) is a valid approximation of the original dynamics.

We now consider two cases, namely β being very small and very large, respectively. For small β , the requirement $u_j = O(\beta)$ for $j > 1$, which means that none of the corresponding particles has vanished, and (2.8) yield

$$\frac{d}{ds}u_0 \sim 2 \quad \text{and} \quad \frac{d}{ds}u_1 \sim -1.$$

The key findings are the following.

1. The particles x_1 and x_{-1} vanish when u_1 reaches a value of order $-\beta$, and that happens at a time of order $s \sim \beta$ or $t \sim 1$.
2. On this time scale the particles x_j , $|j| > 1$ hardly change size due to $|u_j| = O(\beta)$. Therefore, after x_1 and x_{-1} have vanished, we are in the same situation as initially where x_0 is surrounded by small particles $x_j \approx 1$, now indexed by $|j| > 1$.
3. The mass $x_1(0) + x_{-1}(0) = 2$ is almost completely transported to the particle x_0 , and the latter grows linearly in the time t .

With regard to the second and third item we refer to the case of small β as *sequential vanishing* or *mass concentration*.

In the second case, when β is large, we use that $-u_1$ has to become large in order that x_1 vanishes. Due to $u_2 \geq 0$, we have

$$\frac{d}{ds}u_1 \sim -2e^{-u_1} + e^{-u_2} \lesssim -2e^{-u_1},$$

which implies that the time for u_1 to reach a size of order $-\beta$ is of order $s \sim 1$. Moreover, we have

$$\frac{d}{ds}u_2 \sim e^{-u_1} - 2e^{-u_2} + e^{-u_3} \sim -\frac{1}{2}\frac{d}{ds}u_1 - \frac{3}{2}e^{-u_2} + e^{-u_3} \gtrsim -\frac{1}{2}\frac{d}{ds}u_1 - \frac{3}{2}$$

due to $u_2 \geq 0 \geq u_3$, and we conclude that

$$u_2(s) \gtrsim -\frac{1}{2}u_1(s) - \frac{3}{2}s. \quad (2.9)$$

The main observations are now the following.

1. The particles x_1 and x_{-1} vanish at a time of order $s \sim 1$ or $t \sim \frac{1}{\beta}$.
2. The change of the other particles is not known to be smaller. In fact, by (2.9) the change of u_2 is of the same order as the change of $u_1 \sim -\beta$ at times of order $s \sim 1$.
3. Due to (2.9) and the equation for u_0 in (2.8), the mass $x_1(0) = 1$ is split between x_0 and x_j , $j > 1$ with at least half of it (up to an error $1/\beta$) going to x_j , $j > 1$. Similarly, the mass of x_{-1} is split between x_0 and x_j , $j < -1$.

In view of the third item we call this case *mass spreading*.

Our heuristic observations are confirmed by numerical simulations. In fact, the latter even yield a refined picture which suggests that there is a critical value $\beta_c \approx 0.27$ such that for initial data $x_0 = X_0 > 1$ and $x_j = 1$, $|j| > 1$ one of the following three cases holds; compare Figure 2.1.

1. If $\beta > \beta_c$ then for all $X_0 > 1$ mass spreading occurs.
2. For each $\beta < \beta_c$ there is an X_β such that for $X_0 < X_\beta$ mass spreading takes place.
3. For $\beta < \beta_c$ and $X_0 > X_\beta$ mass concentration arises.

Moreover, our numerical results also indicate that in the mass spreading case the particle sizes become disordered over time as indicated in Figure 1.2. Currently, however, we are not able to provide good heuristics for or a suitable notion of being disordered.

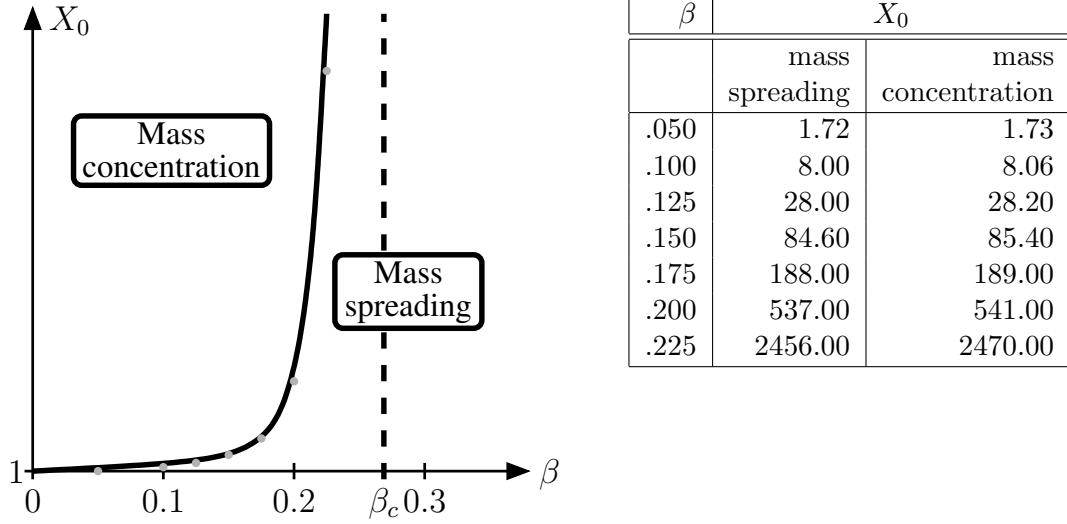


Figure 2.1: “Phase diagram” for the example in Section 2.3. There is a critical β_c , which according to simulations lies between 0.26 and 0.27, such that for $\beta > \beta_c$ always mass spreading occurs. If $\beta < \beta_c$, on the other hand, we have mass spreading for small X_0 and mass concentration for large X_0 . Some numerical values are shown in the table and as gray dots in the diagram.

3 Existence of solutions for initial data with positive density

In this section we prove existence of solutions for suitably distributed initial data which contain no large regions in \mathbb{Z} of vanished or very small particles. We also suppose that initially there are no vanished particles.

Assumption 3.1. All particles in the initial configuration $x = (x_j) \in \ell_+^\infty$ are positive. Moreover, there exist two constants $d > 0$ and $L > 0$ such that \mathbb{Z} can be subdivided into regions of length L with mass density at least d ; that is, there is $k_0 \in \{0, 1, \dots, L-1\}$ such that for

$$R(k) := \{k_0 + kL, \dots, k_0 + (k+1)L - 1\}, \quad k \in \mathbb{Z}$$

we have

$$\frac{1}{L} \sum_{j \in R(k)} x_j \geq d. \quad (3.1)$$

Without loss of generality we assume $k_0 = 0$.

Remark (Traps). Equation (3.1) implies the existence of indices $j_k \in R(k)$ such that $x_{j_k} \geq d$ for all $k \in \mathbb{Z}$. In particular, we have $j_k \rightarrow \pm\infty$ as $k \rightarrow \pm\infty$ and $0 < j_{k+1} - j_k < 2L$. We call the x_{j_k} and j_k *traps*, and given $j \in \mathbb{Z}$ we denote by j_\pm the two nearest traps (unequal to j itself), that is $j_- = \max \{j_k < j\}$ and $j_+ = \min \{j_k > j\}$; obviously, we have

$$j_+ - j_- \leq 4L, \quad j_+ - j \leq 2L, \quad \text{and} \quad j - j_- \leq 2L$$

for all $j \in \mathbb{Z}$. Finally, if x is a particle configuration with traps (j_k) as above, then there is at least one trap among $2L$ consecutive particles. Thus, the existence of traps is up to a factor for L equivalent to the density inequality (3.1).

An example for a particle ensemble that satisfies Assumption 3.1 is a positive $x \in \ell_+^\infty$ such that the one-sided mean particle sizes

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{j=0}^n x_j, \quad \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{j=-n}^0 x_j$$

exist and are positive. To see this, assume for contradiction that there are no appropriate traps, that is, for any $\delta > 0$ the indices (j_k) of particles $x_{j_k} \geq \delta$ either are bounded from above or below or have unbounded distance $j_{k+1} - j_k$ as $k \rightarrow +\infty$ or $k \rightarrow -\infty$. If the latter happens as $k \rightarrow \infty$, we can find an index $k_0 = k_0(\delta)$ such that $j_{k+1} - j_k \geq 1/\delta$ for all $k \geq k_0$, which implies $\#\{j_k : j_{k_0} \leq j_k \leq n\} \leq (n+1)\delta$ for $n > j_{k_0}$. Together with $\#\{j_k : 0 \leq j_k \leq j_{k_0}\} \leq (j_{k_0} + 1)$ and $x_j \leq C$ for all $j \in \mathbb{Z}$ we obtain

$$\begin{aligned} \frac{1}{n+1} \sum_{j=0}^n x_j &= \frac{1}{n+1} \left(\sum_{\substack{j=0, \dots, n \\ x_j \leq \delta}} x_j + \sum_{\substack{j=0, \dots, j_{k_0} \\ x_j > \delta}} x_j + \sum_{\substack{j=j_{k_0}+1, \dots, n \\ x_j > \delta}} x_j \right) \\ &\leq \delta + \frac{C(j_{k_0} + 1)}{n+1} + C\delta \end{aligned}$$

for $n > j_{k_0}$, and sending $n \rightarrow \infty$ gives a contradiction as $\delta > 0$ is arbitrary. The other cases are similar.

The sequential vanishing example in Section 2.3 provides strong evidence that, at least for small β , a condition like Assumption 3.1 is necessary for bounded solutions to exist. Consider an initial configuration that contains infinitely many regions R_j where a large particle of size 1 is surrounded by L_j particles of size ε_j on each side, and assuming that x is a solution for such initial data, rescale time and particle size by means of $y(t) = x(\varepsilon_j^{\beta+1}t)/\varepsilon_j$. Then, y still satisfies equation (2.2), particles of size $x = \varepsilon_j$ are transformed to $y = 1$ and those of size $x = 1$ to $y = 1/\varepsilon_j$. The particles y in R_j thus resemble exactly the sequential vanishing situation in Section 2.3, provided that β and ε_j are small and L_j is sufficiently large so that the particles around the large one in R_j are not influenced by the adjacent regions $R_{j\pm 1}$. In particular, the large particle in R_j gains all the mass of the nearby vanishing ones and its size y grows linearly in time. In terms of x , the large particle grows like

$$\varepsilon_j^{-(\beta+1)} t \varepsilon_j = t \varepsilon_j^{-\beta},$$

which means that we can arrange an increase of order $\varepsilon_j^{-\beta/2}$ in a time of order $\varepsilon_j^{\beta/2}$ if $L_j \varepsilon_j \gg \varepsilon_j^{-\beta/2}$. Hence, by $\varepsilon_j \rightarrow 0$ and choosing L_j appropriately we obtain an initial configuration $x \in \ell^\infty$ so that some particles become arbitrarily large in arbitrarily small time.

3.1 Local-in-time a priori estimates

We now derive key properties of solutions with initial data as in Assumption 3.1. In the following, we let $x: [0, T) \rightarrow \ell^\infty$ be such a solution as in Definition 2.1 and for simplicity of notation we consider times $0 \leq t < T$ without indicating it explicitly. Throughout the paper, we denote by C and c generic constants that depend on β only.

According to Lemma 2.3 the differential equation (2.2) is true for all but the vanishing times $\mathcal{V} = \{\tau_k : k \in \mathbb{Z}\}$, and dropping the non-negative contributions of Δ_σ gives

$$\dot{x}_j(t) \geq -2x_j(t)^{-\beta}$$

whenever $x_j > 0$ and $t \notin \mathcal{V}$. Integrating from t_1 to t_2 , where $0 \leq t_1 < t_2 < \tau_j$, we find

$$x_j(t_2)^{\beta+1} - x_j(t_1)^{\beta+1} \geq -2(\beta+1)(t_2 - t_1) \quad (3.2)$$

and conclude that

$$x_j(t_2) \geq \left(x_j(t_1)^{\beta+1} - 2(\beta+1)(t_2 - t_1) \right)^{\frac{1}{\beta+1}} \quad (3.3)$$

as long as the difference on the right hand side is non-negative. Hence, we obtain the *half-life estimate*

$$x_j(t) \geq \frac{x_j(t_1)}{2} \quad \text{for all} \quad t \leq t_1 + Cx_j(t_1)^{\beta+1}$$

and in particular the *persistence estimate* for traps

$$x_{j_k}(t) \geq \frac{d}{2} \quad \text{for all} \quad t \leq T^* := \min(Cd^{\beta+1}, T).$$

The latter implies

$$\sigma_-(j, x(t)) \geq j_-, \quad \sigma_+(j, x(t)) \leq j_+ \quad \text{for all} \quad t \leq T^*, \quad j \in \mathbb{Z},$$

and therefore the number of different particles that up to time T^* can appear as neighbors of any fixed particle x_j is at most $4L$.

Next, we consider the local masses $M_{m,n}(t) = \sum_{k=m}^n x_k(t)$, $m < n$ as defined in (2.4). Similar to (2.5), we have

$$\frac{d}{dt} M_{m,n}(t) = x_{\sigma_-(m)}(t)^{-\beta} - x_{\sigma_-(\sigma_+(m))}(t)^{-\beta} - x_{\sigma_+(\sigma_-(n))}(t)^{-\beta} + x_{\sigma_+(n)}(t)^{-\beta} \quad (3.4)$$

whenever $M_{m,n}(t) > 0$ and $t \notin \mathcal{V}$, and we infer that

$$\frac{d}{dt} M_{m,n}(t) \leq x_{\sigma_-(m)}(t)^{-\beta} + x_{\sigma_+(n)}(t)^{-\beta} - 2M_{m,n}(t)^{-\beta} \quad (3.5)$$

since $0 < x_{\sigma_-(\sigma_+(m))}(t) \leq M_{m,n}(t)$ and $0 < x_{\sigma_+(\sigma_-(n))}(t) \leq M_{m,n}(t)$. Furthermore, if $j_k < j_l$ are two traps, the persistence estimate and integration of (3.4) provide

$$M_{j_k, j_l}(t_2) \geq M_{j_k, j_l}(t_1) - 2 \left(\frac{d}{2} \right)^{-\beta} (t_2 - t_1) \quad (3.6)$$

and

$$M_{j_k+1, j_l-1}(t_2) \leq M_{j_k+1, j_l-1}(t_1) + 2 \left(\frac{d}{2} \right)^{-\beta} (t_2 - t_1) \quad (3.7)$$

for all $0 \leq t_1 < t_2 < T^*$. From the inequalities (3.6) and (3.7) we draw several conclusions that are essential ingredients of our existence result. First, the lower bound (3.6) implies that the mass density inequality (3.1) in Assumption 3.1 is stable up to time T^* in the following sense.

Lemma 3.2. *For any $q \in \mathbb{N}$, $q > 1$ and any $k \in \mathbb{Z}$ the mass density in the region*

$$R_q(k) := \{qkL, \dots, q(k+1)L - 1\} = R(qk) \cup \dots \cup R(q(k+1) - 1)$$

of length qL satisfies

$$\frac{1}{qL} \sum_{j \in R_q(k)} x_j(t) \geq \left(1 - \frac{1 + C/L}{q} \right) d$$

for all $t \in [0, T^]$.*

Proof. Each of the q regions $R(qk), \dots, R(q(k+1) - 1)$ of Assumption 3.1 contains a trap, and denoting by $k^+ \in R(q(k+1) - 1)$ and $k^- \in R(qk)$ the largest and smallest of these we infer from (3.6) that

$$\sum_{j \in R_q(k)} x_j(t) \geq M_{k^-, k^+}(0) - Cd^{-\beta}t \geq M_{k^-, k^+}(0) - Cd$$

for all $t \leq T^* \leq Cd^{\beta+1}$. Moreover, Assumption 3.1 also implies

$$M_{k^-, k^+}(0) \geq \sum_{n=qk+1}^{q(k+1)-2} \sum_{j \in R(n)} x_j \geq (q-1)Ld,$$

and combining both inequalities finishes the proof. \square

The key point of Lemma 3.2 is the trade-off between the lower bound and the number of regions in the density estimate: by choosing q sufficiently large, we can ensure that the initial bound d is almost preserved.

From the upper mass estimate (3.7) we derive Hölder continuity and boundedness of the particle trajectories.

Lemma 3.3. *We have*

$$|x_j(t_2) - x_j(t_1)| \leq C(L+1)|t_2 - t_1|^{\frac{1}{\beta+1}}$$

for any $j \in \mathbb{Z}$ and all $0 \leq t_1 < t_2 \leq T^*$.

Proof. Given $j \in \mathbb{Z}$, it suffices to consider $t_2 < T^*$ such that $x_j(t_2) > 0$. Using (3.7) with the two nearest traps j_{\pm} and $T^* \leq Cd^{\beta+1}$ we have

$$\begin{aligned} M_{j_-+1, j_+-1}(t_2) &\leq M_{j_-+1, j_+-1}(t_1) + 2(d/2)^{-\beta} T^{*\frac{\beta}{\beta+1}} (t_2 - t_1)^{\frac{1}{\beta+1}} \\ &\leq M_{j_-+1, j_+-1}(t_1) + C(t_2 - t_1)^{\frac{1}{\beta+1}}, \end{aligned}$$

and separating x_j from both mass terms yields

$$x_j(t_2) - x_j(t_1) \leq \sum_{\substack{j_- < k < j_+ \\ k \neq j}} (x_k(t_1) - x_k(t_2)) + C(t_2 - t_1)^{\frac{1}{\beta+1}}. \quad (3.8)$$

We split the sum in (3.8) into large and small particles, respectively, and consider the summation over

$$\begin{aligned} U_L &= \{k \in \{j_- + 1, \dots, j_+ - 1\} \setminus \{j\} : x_k(t_1) > \varepsilon\}, \\ U_S &= \{k \in \{j_- + 1, \dots, j_+ - 1\} \setminus \{j\} : x_k(t_1) \leq \varepsilon\}, \end{aligned}$$

separately, where ε is defined by $\varepsilon^{1+\beta} = 4(\beta+1)(t_2 - t_1)$. For the small particles we simply estimate

$$\sum_{k \in U_S} x_k(t_1) - x_k(t_2) \leq \varepsilon(j_+ - j_-) \leq CL(t_2 - t_1)^{\frac{1}{\beta+1}} \quad (3.9)$$

using $x_k(t_2) \geq 0$, $j_+ - j_- \leq 2L$ and the definition of ε . For $k \in U_L$, on the other hand, we find $x_k(t_1)^{\beta+1} - 2(\beta+1)(t_2 - t_1) > 0$, hence we may use (3.3) and the elementary inequality $(a^{\beta+1} - b^{\beta+1}) \geq (a - b)^{\beta+1}$ for $\beta > 0$ and $0 \leq b \leq a$ to get

$$x_k(t_2) \geq x_k(t_1) - C(t_2 - t_1)^{\frac{1}{\beta+1}}$$

and conclude

$$\sum_{k \in U_L} x_k(t_1) - x_k(t_2) \leq CL(t_2 - t_1)^{\frac{1}{\beta+1}}. \quad (3.10)$$

Combination of (3.8)–(3.10) yields the Hölder estimate in the case $x_j(t_2) \geq x_j(t_1)$, while for $x_j(t_2) < x_j(t_1)$ the result is an immediate consequence of (3.2) and once more the elementary inequality $(a^{\beta+1} - b^{\beta+1}) \geq (a - b)^{\beta+1}$. \square

An upper bound on the particle sizes now follows from either the mass estimate (3.7) or the Hölder inequality.

Corollary 3.4. *We have*

$$x_j(t) \leq M_{j_{-}+1, j_{+}-1}(0) + Cd \quad \text{and} \quad x_j(t) \leq x_j(0) + CLd$$

for all $t \in [0, T^*]$ and any $j \in \mathbb{Z}$.

Finally, we characterize the behavior of vanishing particles. Inequality (3.2) is in case $\tau_j < \infty$ also true for $t_2 \geq \tau_j$ and immediately provides

$$x_j(t) \leq C(\tau_j - t)^{\frac{1}{\beta+1}} \quad \text{for all} \quad 0 \leq t \leq \tau_j, \quad (3.11)$$

which can be understood in two ways. On the one hand, (3.11) is an upper bound on the vanishing particle $x_j(t)$ in terms of $\tau_j - t$, and on the other hand, setting $t = 0$ gives a lower bound on the vanishing time τ_j in terms of $x_j(0)$. In the following two lemmas we prove the opposite assertion by showing first that the size of a vanishing particle is estimated from below by a power law and second that sufficiently small particles do indeed vanish.

Lemma 3.5. *There is a constant $K = K(\beta, d, L)$ such that*

$$x_j(t) \geq K(\tau_j - t)^{\frac{1}{\beta+1}}$$

for all $t \in [0, \tau_j]$ and any $j \in \mathbb{Z}$ with $\tau_j \leq T^*$.

Proof. We prove the claim by contradiction. If it is wrong there exist solutions $x^n : [0, T^n) \rightarrow \ell_+^\infty$, $n \in \mathbb{N}$, such that each has a particle j^n with vanishing time $\tau_{j^n}^n \leq \min(T^n, Cd^{\beta+1})$ and a bad time $t_b^n < \tau_{j^n}^n$ with

$$\frac{x_{j^n}^n(t_b^n)}{(\tau_{j^n}^n - t_b^n)^{\frac{1}{\beta+1}}} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

We may shift the particle index of x^n so that $0 = j_-^n < j^n < j_+^n \leq 4L$ and then select a subsequence of $n \rightarrow \infty$ in which j^n and j_\pm^n are constant. In the following, we restrict our attention to this subsequence of n and to the particles $k = 0, \dots, j_+^n$; for simplicity of notation we drop the index n in the notation.

Rescaling time and size we consider

$$y_k(s) = \frac{x_k(t)}{(\tau_j - t_b)^{\frac{1}{\beta+1}}}, \quad t = \tau_j + (\tau_j - t_b)s,$$

whereby the bad times t_b are translated into $s = -1$ and the vanishing times τ_k of x_k into vanishing times σ_k of y_k ; in particular, we have $\sigma_j = 0$ and deduce

$$y_j(s) > 0 \text{ for all } s \in [-1, 0), \quad y_j(-1) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It is clear that $\frac{d}{ds}y_k(s) = \Delta_\sigma y_k(s)^{-\beta}$ whenever $x_k(t)$ is differentiable at the time t corresponding to s . Moreover, since x_k is Hölder continuous we also obtain

$$|y_k(s_2) - y_k(s_1)| \leq C|s_2 - s_1|^{\frac{1}{\beta+1}}$$

for all $s_1, s_2 \in [-1, 0]$ and $k = 0, \dots, j_+$, where C depends on β and L .

To achieve a contradiction we study the rescaled mass in a neighborhood of j in which the particles vanish as fast as x_j . For that purpose, we let $B \subset \{1, \dots, j_+ - 1\}$ be the largest set of consecutive indices containing j such that

$$\liminf_{n \rightarrow \infty} y_k(-1) = 0 \quad \text{for all } k \in B;$$

writing $B = \{m+1, \dots, l-1\}$ and restricting ourselves to a subsequence of n we may assume that the lower limits are attained as limits and that $y_m(-1) \geq c$, $y_l(-1) \geq c$ for all n and some constant $c > 0$. Hölder continuity then implies

$$y_k(s) \geq c - C\delta^{\frac{1}{\beta+1}} \geq \frac{c}{2} \quad \text{for } k = m, l,$$

$s \in [-1, -1 + \delta)$ and sufficiently small $\delta > 0$, whereas the rescaled mass $M(s) = \sum_{k \in B} y_k(s)$ satisfies

$$M(s) \leq M(-1) + 4LC\delta^{\frac{1}{\beta+1}} \leq 2^{-\frac{1}{\beta}} \frac{c}{2}$$

for $s \in [-1, -1 + \delta)$, sufficiently small $\delta \in (0, 1)$ and all sufficiently large n . Choosing $\delta = \delta(\beta, L, c)$ and $n = n(\beta, L, c, \delta)$ so that these two inequalities are true, we obtain

$$\frac{d}{ds} M(s) = \sum_{\substack{k \in B \\ y_k(s) > 0}} \Delta_\sigma y_k(s) \leq y_m(s)^{-\beta} + y_l(s)^{-\beta} - 2M(s)^{-\beta} \leq -M(s)^{-\beta}$$

for all $s \in (-1, -1 + \delta) \setminus \{\sigma_k : k \in B\}$ and therefore

$$0 < y_j(s)^{\beta+1} \leq M(s)^{\beta+1} \leq M(-1)^{\beta+1} - (\beta+1)\delta.$$

The contradiction now follows from increasing n until $M(-1)^{\beta+1} < (\beta+1)\delta$. \square

Lemma 3.6. *For any $t_0 \in (0, T^*)$ there is a constant $\eta = \eta(\beta, d, L, t_0)$ such that $x_j(t) \leq \eta$ for some $t \in [0, t_0]$ and $j \in \mathbb{Z}$ implies $\tau_j \leq t + (x_j(t)/K)^{\beta+1}$, where $K = K(\beta, d, L)$ is the constant from Lemma 3.5.*

Proof. Lemma 3.5 yields $\tau_j \leq t + (x_j(t)/K)^{\beta+1}$ provided that $\tau_j \leq T^*$; hence, we only have to prove the latter inequality in the case $x_j(t) > 0$. Furthermore, by a shift in time it is sufficient to consider $t = 0$.

Following the same idea as in the proof of Lemma 3.5, we assume for contradiction that the claim is wrong. Then there are a sequence $\eta_n \rightarrow 0$ as $n \rightarrow \infty$ and solutions $x^n : [0, T^n)$ such that $T^n \geq t_0$ and

$$x_{j_n}^n(0) \leq \eta_n, \quad x_{j_n}^n(t) > 0 \text{ for all } t < \min(T^n, Cd^{\beta+1})$$

for some particle j^n . Again, by shifting the particle index we select a subsequence of $n \rightarrow \infty$ such that $j_-^n = 0$, j^n and j_+^n are constant, drop the index n in the notation, and restrict our attention to the particles $k = 0, \dots, j_+$.

We now rescale time and particle sizes according to

$$y_k(s) = \frac{x_k(t)}{\eta}, \quad t = \eta^{\beta+1} s$$

and denote by $B = \{m+1, \dots, l-1\} \subset \{0, \dots, j_+\}$ the largest set of consecutive particles containing j such that

$$\liminf_{n \rightarrow \infty} y_k(0) = \alpha_k \in [0, \infty) \quad \text{for all } k \in B;$$

restriction to a subsequence of n allows us to assume that the lower limits are attained as limits and that for $k = m, l$ we have $y_k(0) \rightarrow \infty$ as $n \rightarrow \infty$. As before, Hölder continuity implies

$$y_k(s) \geq c - Cs^{\frac{1}{\beta+1}} \quad \text{for } k = m, l, \quad c = \min_{k=m,l} y_k(0),$$

and $s \in [0, S]$ where $S = T\eta^{-(\beta+1)}$, while the rescaled mass $M(s) = \sum_{k \in B} y_k(s)$ satisfies

$$M(s) \leq M(0) + 4LCs^{\frac{1}{\beta+1}}.$$

Since $S \rightarrow \infty$, $c \rightarrow \infty$ and $M(0) \rightarrow \sum_{k \in B} \alpha_k$ as $n \rightarrow \infty$, we can for any $\delta > 0$ find an n such that $\delta < S$ and

$$y_k(s) \geq c - C\delta^{\frac{1}{\beta+1}} \geq 2^{1/\beta} (M(0) + 4LC\delta^{\frac{1}{\beta+1}}) \geq 2^{1/\beta} M(s)$$

for $k = m, l$ and all $s \in [0, \delta]$. Hence, we obtain

$$\frac{d}{ds} M(s) \leq y_m(s)^{-\beta} + y_l(s)^{-\beta} - 2M(s)^{-\beta} \leq -M(s)^{-\beta}$$

for all $s \in (0, \delta) \setminus \{\eta^{-(\beta+1)}\tau_k : k \in B\}$ and

$$0 < y_j(\delta)^{\beta+1} \leq M(\delta)^{\beta+1} \leq M(0)^{\beta+1} - (\beta+1)\delta.$$

Choosing δ and n sufficiently large gives a contradiction. \square

Remark. If $\tau_j > T^*$, then $x_j(t) \geq c > 0$ for all $t \in [0, T^*]$. This and Lemma 3.5 if $\tau_j \leq T^*$ imply that every $x_j^{-\beta} \chi_{\{x_j > 0\}}$ is integrable in $[0, T^*]$. Hence, for initial data as in Assumption 3.1 any solution in $[0, T^*)$ to the differential equation (2.2) as in the remark after Lemma 2.3 is a solution according to Definition 2.1.

3.2 Existence of solutions

We now prove existence of a solution to (2.2) by truncating the initial data at a particle index n and sending $n \rightarrow \infty$. To this end, given $x^\infty \in \ell_+^\infty$ as in Assumption 3.1, we first look for a solution of the finite dimensional initial value problem

$$\dot{x}_j^n(t) = \Delta_\sigma x_j^n(t)^{-\beta} \quad \text{if } t > 0 \quad \text{and} \quad x_j^n(0) = x_j^\infty \quad \text{for } j = -n, \dots, n. \quad (3.12)$$

Recall that (3.12) is well-defined by definition of Δ_σ , in which contributions from beyond the two outermost living particles are dropped.

By a solution to (3.12) we mean a continuous $x^n : [0, \infty) \rightarrow X^n$, where

$$X^n = \{x = (x_j) \in \mathbb{R}^{2n+1} : 0 \leq x_j \leq C \text{ for } j = -n, \dots, n \text{ and some constant } C > 0\},$$

such that x^n attains the initial data in (3.12) and has the following property: For each $j = -n, \dots, n$ there exists a vanishing time $\tau_j^n \in (0, \infty]$ as defined before, and between $t = 0$, consecutive vanishing times and $t = \infty$ each x_j^n is continuously differentiable and satisfies the differential equation $\dot{x}_j^n = \Delta_\sigma(x_j^n)^{-\beta}$. In short, x^n solves the same differential equation as in the full problem, but with truncated initial data, and consequently, the results of Section 3.1 are true for x_j^n provided that n is larger than the particles involved. In particular, we have

1. persistence of traps

$$x_{j_k}^n(t) \geq \frac{d}{2}$$

for all $t \in [0, T^*]$ and j_k, n such that $n > |j_k|$;

2. the upper bound

$$x_j^n(t) \leq C$$

for $t \in [0, T^*]$ and all j, n such that $n > |j_\pm|$ where C depends on β, d, L and $\sup_{j \in \mathbb{Z}} x_j^\infty$;

3. the Hölder estimate

$$|x_j^n(t_2) - x_j^n(t_1)| \leq C|t_2 - t_1|^{\frac{1}{\beta+1}}$$

for all $0 \leq t_1 < t_2 \leq T^*$ and all j, n such that $n > |j_\pm|$, where $C = C(\beta, L)$; and

4. the vanishing estimate

$$x_j^n(t) \leq \eta \text{ for some } t \leq t_0 \quad \implies \quad \tau_j^n \leq t + Cx_j^n(t)^{\beta+1} \leq T^*$$

for $t_0 \in (0, T^*)$, sufficiently small $\eta = \eta(\beta, d, L, t_0)$ and all j, n such that $n > |j_\pm|$, which in turn implies

$$x_j^n(t) \geq C(\tau_j^n - t)^{\frac{1}{\beta+1}}$$

for $t \in [0, \tau_j^n]$, where $C = C(\beta, d, L)$.

Furthermore, equation (3.5) for the total mass $M_{-n,n}^n(t) = \sum_{j=-n}^n x_j(t)$ of the truncated system provides

$$x_j^n(t) \leq M_{-n,n}^n(t) \leq M_{-n,n}^n(0), \quad (3.13)$$

which means that *all* x_j , $j = -n, \dots, n$ are bounded for fixed n and not only those between two traps.

Lemma 3.7 (Existence for the truncated system). *For any $n \in \mathbb{N}$ and initial data $x^n(0) = (x_j^n(0))_{j=-n, \dots, n}$ such that $0 < x_j^n(0) < \infty$ there is a solution to (3.12).*

Proof. Since (3.12) is an ordinary differential equation with locally Lipschitz continuous right hand side in $(0, \infty)^{2n+1}$, Picard's Theorem yields a time $\theta_0 > 0$ and a unique solution $x^n \in C^1([0, \theta_0]; (0, \infty)^{2n+1})$. By inequality (3.13), the x_j^n are bounded, hence the solution can be extended either indefinitely or until a time θ_1 at which one or more of the x_j^n vanish. In the latter case, we have $x_j^n \in C^1((0, \theta_1)) \cap C^0([0, \theta_1])$ for all $j = -n, \dots, n$, and the evolution can be continued by restriction to the living particles at time θ_1 . By iteration we thus obtain a solution, which after at most $2n + 1$ steps is identically 0. \square

Theorem 3.8 (Local existence). *Given initial data as in Assumption 3.1 and $T_1 < T^*$, where $T^* = Cd^{\beta+1}$ as in the persistence estimate, there exist a subsequence of (x^n) , not relabeled, and a solution $x: [0, T_1] \rightarrow \ell^\infty$ to (2.2) such that $x_j^n \rightarrow x_j$ uniformly in $[0, T_1]$ for any $j \in \mathbb{Z}$.*

Proof. Let x^n , $n \in \mathbb{N}$ be the solution from Lemma 3.7 and set $x_j^n \equiv 0$ for $|j| > n$. Then for any fixed $j \in \mathbb{Z}$ the sequence $(x_j^n)_{n \in \mathbb{N}}$ is uniformly bounded in $C^{0,1/(\beta+1)}([0, T^*], [0, \infty))$ and by the Arzelà-Ascoli Theorem compact in $C^{0,\alpha}([0, T^*])$ for $0 \leq \alpha < 1/(\beta+1)$. Without relabeling we may thus extract a subsequence of $n \rightarrow \infty$ for $j = 0, \pm 1, \pm 2, \dots$ successively and take a diagonal sequence to obtain $x: [0, T^*] \rightarrow \ell_+^\infty$ such that $x_j^n \rightarrow x_j$ in $C^{0,\alpha}([0, T^*])$ and $x_j \in C^{0,1/(\beta+1)}([0, T^*])$ for any $j \in \mathbb{Z}$. By a further subsequence and diagonal argument we may assume that each sequence $(\tau_j^n)_{n \in \mathbb{N}}$ converges to some $\theta_j \in (0, \infty]$ as $n \rightarrow \infty$.

We aim to prove that x is a solution to (2.2) by passing to the limit $n \rightarrow \infty$ in the integral equation

$$x_j^n(t_2) - x_j^n(t_1) = \int_{t_1}^{t_2} \Delta_\sigma x_j^n(s)^{-\beta} ds$$

for $0 \leq t_1 < t_2 \leq T_1$ and any $j \in \mathbb{Z}$. To this end, we fix $\eta = \eta(\beta, d, L, T_1) > 0$ such that the vanishing estimate holds with $t_0 = T_1$ and infer that for each j and $n > |j_\pm|$ we have either

$$\tau_j^n > T^* \quad \text{and} \quad x_j^n(t) > \eta \text{ for all } t \in [0, T_1] \quad (3.14)$$

or

$$\tau_j^n \leq T^* \quad \text{and} \quad x_j^n(t) \geq C(\tau_j^n - t)^{\frac{1}{\beta+1}} \quad (3.15)$$

Hence, we may choose another subsequence of $n \rightarrow \infty$ such that for each j exactly one of (3.14), (3.15) holds for all large n . In the first case we trivially have $x_j^n \geq \eta$ for all large n and obtain

$$(x_j^n)^{-\beta} \chi_{\{x_j^n > 0\}} \rightarrow x_j^{-\beta} \chi_{\{x_j > 0\}} \quad \text{in } L^1(0, T_1).$$

In the second case we find that $\theta_j \leq T^*$ is the unique vanishing time of x_j in $[0, T^*]$, because $0 = x_j^n(t) \rightarrow x_j(t)$ for $t \geq \theta_j$ and

$$x_j(t) \geq \lim_{n \rightarrow \infty} C(\tau_j^n - t)^{\frac{1}{\beta+1}} = C(\theta_j - t)^{\frac{1}{\beta+1}} > 0$$

for $t < \theta_j$. Furthermore, we obtain convergence of $(x_j^n)^{-\beta} \chi_{\{x_j^n > 0\}}$ in $L^1(0, T_1)$ from pointwise convergence of $\chi_{\{x_j^n > 0\}}$ to $\chi_{\{x_j > 0\}}$, the upper bound

$$x_j^n(t)^{-\beta} \chi_{\{x_j^n > 0\}} \leq C(\tau_j^n - t)^{-\frac{\beta}{\beta+1}} \chi_{\{t < \tau_j^n\}},$$

and the generalized Dominated Convergence Theorem. Finally, the persistence of traps and uniqueness of vanishing times imply that each x_j has no more than $4L$ different neighbors in the time interval $[0, T_1]$, and convergence of $\Delta_\sigma(x_j^n)^{-\beta}$ in $L^1(0, T_1)$ follows. \square

We conclude this section with our global existence result, which follows from combining Theorem 3.8 and the lower density estimate in Lemma 3.2.

Theorem 3.9 (Global existence). *For initial data as in Assumption 3.1 there exists a solution $x: [0, \infty) \rightarrow \ell_+^\infty$ to (2.2), which has the following properties:*

1. *for any $T > 0$ there is a constant $C = C(\beta, d, L, T)$ such that $x_j(t) \leq x_j(0) + C$ for all $t \in [0, T]$ and all $j \in \mathbb{Z}$;*
2. *for any $T > 0$ each x_j is Hölder continuous in $[0, T]$ with exponent $1/(\beta + 1)$ and $\|x_j\|_{C^{0,1/(\beta+1)}([0,T])} \leq C = C(\beta, d, L, T)$;*
3. *if $\tau_j < \infty$ then $x_j(t) \leq C(\tau_j - t)^{1/(\beta+1)}$ for all $t \in [0, \tau_j]$, where $C = C(\beta)$;*
4. *for any $T > 0$ there is a constant $K = K(\beta, d, L, T)$ such that $\tau_j < T$ implies $x_j(t) \geq K(\tau_j - t)^{1/(\beta+1)}$ for all $t \in [0, \tau_j]$ and for any $j \in \mathbb{Z}$.*

Proof. Set $T_1 = T^*/2 = Cd^{\beta+1}/2$, where $C = C(\beta)$, and let $x: [0, T_1] \rightarrow \ell_+^\infty$ be a solution from Theorem 3.8. For $\varepsilon_1 \in (0, 1)$ to be chosen below, Lemma 3.2 provides a sufficiently large $q_1 = q_1(\beta, L, \varepsilon_1)$ so that $x(T_1)$ satisfies the lower density estimate in Assumption 3.1 with $R(k)$ replaced by $R_{q_1}(k)$, L replaced by $L_1 := q_1 L$ and d replaced by $d_1 := (1 - \varepsilon_1)d$. Therefore, we may apply the local existence result once more to extend the solution up to the time $T_2 = T_1 + Cd_1^{\beta+1}/2$. Moreover, the a priori estimates are true in the whole interval $[0, T_2)$, once the constants that depend on the traps are adapted appropriately; this, of course, makes them time-dependent.

With $0 < \varepsilon_n < 1$, $n \in \mathbb{N}$ we may iterate the preceding argument to find

$$d_n = d \prod_{k=1}^n (1 - \varepsilon_k), \quad L_n = L \prod_{k=1}^n q_k, \quad T_n = T_1 + \frac{C}{2} \sum_{k=1}^{n-1} d_k^{\beta+1},$$

and a solution up to any of the times T_n . Choosing ε_n such that $d_n \rightarrow 0$ sufficiently slowly, we can arrange that $T_n \rightarrow \infty$ as $n \rightarrow \infty$. Finally, since any $T > 0$ is reached by the sequence (T_n) after finitely many steps, the claimed estimates follow once more from adapting the constants of the local ones. \square

4 Non-uniqueness

In this section we show that solutions to (2.2) are in general not uniquely determined by their initial data. To this end, we consider once more truncated problems, this time with two slightly different sequences of initial particles that have the same limit but yield two different solutions. More precisely, for $N \in \mathbb{N}$ we consider initial data $\alpha^N \in \ell_+^\infty$ containing particles

$$\alpha_j^N = 0 \quad \text{for } |j| > 3N, \quad \alpha_j^N = 1 \quad \text{for } -3N \leq j < 0, \quad \alpha_{3N}^N = 1,$$

and

$$\alpha_{3j}^N = 1, \quad \alpha_{3j+1}^N = R_{j,1}^N, \quad \alpha_{3j+2}^N = R_{j,2}^N \quad \text{for } j = 0, 1, \dots, N-1,$$

where $(R_{j,p}^N)_{j=0,\dots,N-1}$, $p = 1, 2$ are two decreasing sequences of positive numbers to be chosen; compare Figure 1.3. By x^N we denote the solution to (2.2) with initial data

$$x_j^N(0) = \begin{cases} \alpha_j^N & \text{if } j \neq 3N, \\ \alpha_j^N + \varepsilon^N & \text{if } j = 3N, \end{cases} \quad (4.1)$$

where (ε^N) is another sequence to be specified; in case $\varepsilon^N = 0$ we write \bar{x}^N .

We will prove the following result.

Theorem 4.1. *Let*

$$\beta \geq \beta_* := \frac{\ln 4 - \ln 3}{\ln 3 - \ln 2} \iff 2^{\frac{1}{\beta+1}} \leq \frac{3}{2}. \quad (4.2)$$

Then there are sequences $(R_{j,p}^N)_{j=0,\dots,N-1}$, $N \in \mathbb{N}$, $p = 1, 2$ and $(\varepsilon^N)_{N \in \mathbb{N}}$ with the following properties.

1. *For any $j = 0, 1, \dots$ and $p = 1, 2$ the sequence $(R_{j,p}^N)$ converges to some $R_{j,p} > 0$ as $N \rightarrow \infty$.*
2. *There are a time $T > 0$, functions $\bar{x}: [0, T] \rightarrow \ell_+^\infty$, $x: [0, T] \rightarrow \ell_+^\infty$, and a subsequence of $N \rightarrow \infty$, not relabeled, such that for all $j \in \mathbb{Z}$ we have $\bar{x}_j^N \rightarrow \bar{x}_j$ and $x_j^N \rightarrow x_j$ in $C([0, T])$ as $N \rightarrow \infty$.*
3. *Both, \bar{x} and x solve (2.2) in $[0, T)$ with initial data $\bar{x}(0) = x(0) = \alpha$, where $\alpha_j = 1$ for $j < 0$ and $\alpha_{3j} = 1$, $\alpha_{3j+1} = R_{j,1}$, $\alpha_{3j+2} = R_{j,2}$ for $j \geq 0$, but we have $\bar{x} \neq x$.*

Remark. Inequality (4.2) is a technical condition that we use to study in a simple way the phase portrait of an equation associated with two adjacent small particles. We believe that it can be removed or at least improved by more careful considerations; see the proof of Lemma 4.7 and the remark thereafter.

4.1 Overview of the proof

The proof of Theorem 4.1 is rather technical, so we first outline the main ideas. To this end, we denote as above by \bar{x}^N and x^N two solutions that have initial data as in (4.1) with $\varepsilon^N = 0$ and $\varepsilon^N > 0$ to be specified, respectively; for simplicity of notation we drop the upper index N where possible. Existence of \bar{x} and x as well as convergence to solutions of (2.2) along a subsequence of $N \rightarrow \infty$ follows from the a priori estimates and compactness arguments in Section 3. Furthermore, since we will chose $\varepsilon^N \rightarrow 0$ as $N \rightarrow \infty$, both limit solutions will have the same initial data.

Theorem 4.1 will follow from the fact that the differences

$$|x_j(t) - \bar{x}_j(t)| \quad (4.3)$$

are uniformly positive for some fixed index $j = 3N_*$, some $t > 0$ and all sufficiently large N . To prove this fact, we will on the one hand derive detailed asymptotic formulas for (4.3) in case $j = 3N_*, 3(N_* + 1), \dots, 3N$. On the other hand, we will require a precise description of solutions near vanishing times, because many small particles between $3N_*$ and $3N$ will have vanished at any positive time $t > 0$.

For the analysis to come we require that non-adjacent small particles of \bar{x} and x vanish at well-separated times. Moreover, the details will become simpler if we choose the sequences $(R_{j,p}^N)$ such that the two adjacent small particles of one solution, say \bar{x}_{3j+1} and \bar{x}_{3j+2} vanish simultaneously for each $j = N_*, \dots, N - 1$. In Section 4.2 we show that such a choice is possible, and to that aim it is necessary to study the dependence of solutions on $(R_{j,p}^N)$. The main technical ingredients here are a continuity result for vanishing times in Lemma 4.3 and estimates of the vanishing times in a local problem where two small particles are surrounded by two large ones in Lemma 4.4. The existence of $(R_{j,p}^N)$ with the desired properties then follows from a degree theory type argument in Proposition 4.5.

As a consequence of Section 4.2, we obtain initial data such that for $j = N_*, \dots, N - 1$ the particles \bar{x}_{3j+p} and x_{3j+p} have finite vanishing times $\bar{\tau}_{3j+p}$ and τ_{3j+p} , respectively. Furthermore, $T_j = \max\{\bar{\tau}_{3j+1}, \tau_{3j+1}, \tau_{3j+2}\}$ defines a sequence such that all small particles to the right of $3j$ have vanished at time T_j whereas all small particles to the left of $3j$ have not. The key step of our proof is to show that the most important contribution to $x - \bar{x}$ at time T_{j-1} is $x_{3(j-1)}(T_{j-1}) - \bar{x}_{3(j-1)}(T_{j-1})$ and stems from $x_{3j}(T_j) - \bar{x}_{3j}(T_j)$. Ultimately, this contribution originates from the initial difference $x_{3N} - \bar{x}_{3N} = \varepsilon$, is iteratively propagated through the solutions and amplified by the vanishing of the small particles.

To illustrate the iterative step from T_j to T_{j-1} let us assume that at time T_j both \bar{x} and x are equal except for the particle $3j$ and that we have the simplified situation

$$\begin{aligned} \bar{x}_{3k}(T_j) = x_{3k}(T_j) = 1, \quad \bar{x}_{3k+p}(T_j) = x_{3k+p}(T_j) = R_{k,p} \quad & \text{for } k < j, \\ \bar{x}_{3k}(T_j) = x_{3k}(T_j) = 1, \quad \bar{x}_{3k+p}(T_j) = x_{3k+p}(T_j) = 0 \quad & \text{for } k > j \end{aligned} \quad (4.4)$$

and

$$\bar{x}_{3j}(T_j) = 1, \quad x_{3j}(T_j) = 1 + \delta \quad (4.5)$$

for some small $\delta > 0$. Then we obtain in Theorem 4.9 that the difference $|\bar{x}_{3j}(T_j) - x_{3j}(T_j)| = \delta$ is transported to the particles $3(j - 1)$ where we have

$$|\bar{x}_{3(j-1)}(T_{j-1}) - x_{3(j-1)}(T_{j-1})| \sim \left(R_{j-1}^{4\beta+1} \delta\right)^{\frac{1}{3\beta+1}} \quad (4.6)$$

with R_{j-1} comparable to both $R_{j-1,p}$, $p = 1, 2$. Formula (4.6) quantifies the amplification effect provided that δ is sufficiently small compared to R_{j-1} and we use it in Section 4.5 to choose initial perturbations ε^N for $N > N_*$ such that

$$\delta_N = \varepsilon^N, \quad \delta_{j-1} \sim \left(R_{j-1}^{4\beta+1} \delta_j\right)^{\frac{1}{3\beta+1}} \quad \text{for } j = N_* + 1, \dots, N$$

indeed leads to an iterative amplification. More precisely, we show that we can prescribe $\delta_{N_*} > 0$ sufficiently small and independent of N such that there is $(\varepsilon_N)_{N > N_*}$ with the properties that

$$\varepsilon^N \sim \delta_{N_*}^{(3\beta+1)^{N-N_*}}$$

and

$$|x_{3N_*}^N(T_{N_*}^N) - \bar{x}_{3N_*}^N(T_{N_*}^N)| = \delta_{N_*}$$

for all large N . Then, taking the limit $N \rightarrow \infty$ yields the non-uniqueness result.

The main part of the proof is the analysis of one iteration providing (4.6). It is carried out in Section 4.3, where we study the local problem of two small particles surrounded by a large one on each side, and in Section 4.4, where we deal with the full particle system. First, considering again the setting (4.4)–(4.5), we compute the asymptotic formulas

$$\bar{x}_{3(j-1)+p}(t) \sim (\bar{\tau}_{3(j-1)+1} - t)^{\frac{1}{\beta+1}} \quad \text{as} \quad t \rightarrow \bar{\tau}_{3(j-1)+1}$$

by a straightforward local analysis in Lemma 4.6. Then, since $x(T_j)$ and $\bar{x}(T_j)$ differ by an amount of order δ it is natural to linearize

$$D_p(t) = x_{3(j-1)+p}(t) - \bar{x}_{3(j-1)+p}(t) \sim \delta \phi_p(t) + \text{higher order terms} \quad (4.7)$$

for $t > T_j$ and $p = 1, 2$. This is done in Proposition 4.15, where we in particular obtain that ϕ_p becomes singular near the vanishing times of the particles $3(j-1)+p$ and that its most singular contribution is of order

$$(\bar{\tau}_{3(j-1)+1} - t)^{-\frac{3\beta}{\beta+1}}. \quad (4.8)$$

Hence, by (4.7) and (4.8) the linearization is valid only up to times such that

$$(\bar{\tau}_{3(j-1)+1} - t)^{\frac{3\beta+1}{\beta+1}} \sim \delta,$$

where $\bar{x}_{3(j-1)+p}$ and ϕ_p become comparable, and for later times we have to study the full nonlinear system with suitable matching conditions in order to approximate $x_{3(j-1)+p}$. The local version of the latter problem is analyzed in Lemma 4.7 and the consequences for the vanishing times in the particle system are given in Corollary 4.17. In the end, the approximations for $\bar{x}_{3(j-1)+p}$, $x_{3(j-1)+p}$ as well as $\bar{\tau}_{3(j-1)+p}$, $\tau_{3(j-1)+p}$ enable us to calculate (4.6) in Lemma 4.20, Corollary 4.21 and Proposition 4.22.

Besides the key steps just described, Section 4.4 contains several lemmas that provide estimates for particles other than $3(j-1)+p$ and their differences, which are not necessarily zero as in the simplified setting above. The contributions of these particles, however, are small compared to (4.6), and their estimates usually follow from straightforward computations and standard ODE arguments such as Gronwall's inequality.

4.2 Choice of the small particles

With $0 < \gamma < 1/3$ to be fixed later we set

$$R_j = \gamma^j, \quad j \in \mathbb{N}, \quad (4.9)$$

and let $(R_{j,1}^N)_{j=0,\dots,N-1}$, $(R_{j,2}^N)_{j=0,\dots,N-1}$ satisfy

$$\frac{1}{2}R_j \leq R_{j,p}^N \leq \frac{3}{2}R_j \quad (4.10)$$

for $j = 0, \dots, N-1$, $N \in \mathbb{N}$ and $p = 1, 2$. Moreover, we assume that $(\varepsilon^N)_{N \in \mathbb{N}}$ is a sequence such that

$$|\varepsilon^N| < \frac{1}{8} \quad \text{and} \quad \lim_{N \rightarrow \infty} \varepsilon^N = 0. \quad (4.11)$$

Then, the arguments that led to existence of solutions apply here with $d = 3/4$ and $L = 3$, and for later reference we summarize the corresponding results as follows. Recall that C, c denote generic constants that depend only on β , if not stated otherwise.

Proposition 4.2. *There is a time $T^* > 0$ such that for any $N \in \mathbb{N}$, any sequences $(R_{j,p}^N)$ as in (4.10), and any (ε^N) as in (4.11) there exists a solution $x^N: [0, \infty) \rightarrow X^{6N+1}$ to (2.2) with initial data (4.1) and has the following properties.*

1. *For all $t \in [0, T^*]$ and all initially large particles, that is, $k = -3N, \dots, 0, k = 3j$ with $j = 0, \dots, N-1$, and $k = 3N$, we have $x_k^N(t) \geq 1/2$.*
2. *There is a unique vanishing time $\tau_k^N \in (0, T^*] \cup \{\infty\}$ for each particle $k = -3N, \dots, 3N$ and there are two constants C, c such that if $\tau_k^N < \infty$ for some $|k| < 3N$ we have*

$$x_k^N(t) \leq C(\tau_k^N - t)^{\frac{1}{\beta+1}} \quad (4.12)$$

and

$$x_k^N(t) \geq c(\tau_k^N - t)^{\frac{1}{\beta+1}} \quad (4.13)$$

for all $t \in [0, \tau_k^N]$.

3. *For any $t_0 \in (0, T^*)$ there is $\eta = \eta(\beta, t_0)$ such that $x_k(t_0) \leq \eta$ for some $|k| < 3N$ implies $\tau_k \leq t_0 + (x_k(t_0)/c)^{\beta+1}$.*

The existence of $(R_{j,p}^N)$ such that adjacent small particles of the solution \bar{x} vanish simultaneously follows from the continuity of vanishing times and their asymptotic behavior as particle sizes become locally small. We consider these issues in the following two lemmas.

Lemma 4.3 (Continuity of vanishing times). *There is $N_* > 0$ such that for any $N > N_*$ and any solution x with (R_j) , $(R_{j,p}^N)$ and (ε^N) as in (4.9)–(4.11) we have*

$$cR_j^{\beta+1} \leq \tau_{3j+p}^N \leq CR_j^{\beta+1} < T^* \quad \text{for} \quad j = N_*, \dots, N-1, \quad p = 1, 2.$$

Moreover, the τ_{3j+p}^N change continuously with $((R_{m,1}^N)_{m=N_*, \dots, N-1}, (R_{m,2}^N)_{m=N_*, \dots, N-1}) \in \prod_{k=N_*}^{N-1} (\frac{1}{2}R_k, \frac{3}{2}R_k)^2$.

Proof. The first claim is an immediate consequence of (4.9), (4.10) and Proposition 4.2. To prove continuity, we fix $N > N_*$ and let x be a solution for initial data (α_j^N) with $(R_{j,p}^N)$ and ε^N as in (4.9)–(4.11). Moreover, we denote by y a solution for nearby initial data where $R_{j,p}^N$ is replaced by $R_{j,p}^N + \delta_{j,p}$ with $|\delta_{j,p}| \leq \delta$ for $j = N_*, \dots, N-1$ and $p = 1, 2$. We write τ and θ for the vanishing times of x and y , respectively, and order (τ_{3j+p}) by size via

$$0 < \tau_{3j_1+p_1} \leq \tau_{3j_2+p_2} \leq \dots \tau_{3j_{N-N_*}+p_{N-N_*}},$$

where $(j_m)_{m=1, \dots, N-N_*} \subset \{N_*, \dots, N-1\}$ and $(p_m)_{m=1, \dots, N-N_*} \subset \{1, 2\}^{N-N_*}$.

Given small $\varepsilon > 0$, our first step is to show by a continuation argument that the differences $D_k(t) = y_k(t) - x_k(t)$, $k = -3N, \dots, 3N$ remain arbitrarily small for $t \leq \tau_{3j_1+p_1} - \varepsilon$, provided that we choose δ sufficiently small. Indeed, Proposition 4.2(1 and 2) implies

$$x_k(t) \geq c\varepsilon^{\frac{1}{\beta+1}}$$

for all $|k| \leq 3N$ and $t \leq \tau_{3j_1+p_1} - \varepsilon$, and assuming that $y_k(t) \geq c\varepsilon^{1/(\beta+1)}/2$ we compute

$$\left| y_k(t)^{-\beta} - x_k(t)^{-\beta} \right| = \beta \left| \int_{x_k(t)}^{y_k(t)} s^{-(\beta+1)} ds \right| \leq C \frac{1}{\varepsilon} |D_k(t)|.$$

Using the differential equation for x_k and y_k , we find

$$\frac{d}{dt} \|D_k(t)\|_\infty \leq C \frac{1}{\varepsilon} \|D_k(t)\|_\infty,$$

and by integration we obtain

$$|D_k(t)| \leq \delta \exp \left(C \frac{1}{\varepsilon} (\tau_{3j_1+p_1} - \varepsilon) \right) \quad (4.14)$$

for all $|k| \leq 3N$ and $t \leq \tau_{3j_1+p_1} - \varepsilon$ such that $y_k(t) \geq c\varepsilon^{1/(\beta+1)}/2$. By choosing δ sufficiently small, the latter inequality holds up to some positive t and the right hand side of (4.14) can be made arbitrarily small. Thus, for any $0 < \omega < c\varepsilon^{1/(\beta+1)}/4$ we obtain

$$|D_k(t)| \leq \omega \quad (4.15)$$

and

$$y_k(t) \geq x_k(t) - \omega \geq \frac{3c}{4} \varepsilon^{\frac{1}{\beta+1}} > \frac{c}{2} \varepsilon^{\frac{1}{\beta+1}} \quad (4.16)$$

for all $|k| \leq 3N$, all small δ depending on ε and ω , and all t such that $y_k(t) \geq c\varepsilon^{1/(\beta+1)}/2$. By (4.16) the above reasoning extends to all times $t \leq \tau_{3j_1+p_1} - \varepsilon$. We conclude that $\theta_k \geq \tau_{3j_1+p_1} - \varepsilon$, and moreover we can use (4.15) and (4.12) to transfer an upper bound from $x_{3j_1+p_1}$ to $y_{3j_1+p_1}$, namely

$$y_{3j_1+p_1}(\tau_{3j_1+p_1} - \varepsilon) \leq x_{3j_1+p_1}(\tau_{3j_1+p_1} - \varepsilon) + \omega \leq C\varepsilon^{\frac{1}{\beta+1}} + \omega \leq C\varepsilon^{\frac{1}{\beta+1}}.$$

Proposition 4.2(3) with $t_0 = \tau_{3j_1+p_1} - \varepsilon$ then yields $\theta_{3j_1+p_1} \leq \tau_{3j_1+p_1} + C\varepsilon$.

Let now $s \in \{2, \dots, N - N_*\}$ be the index such that

$$\tau_{3j_1+p_1} = \tau_{3j_2+p_2} = \dots = \tau_{3j_{s-1}+p_{s-1}} < \tau_{3j_s+p_s}.$$

Estimating $\theta_{3j_m+p_m}$, $m = 1, \dots, s-1$ as above, we can arrange that for given $\varepsilon > 0$ we have

$$|\theta_{3j_m+p_m} - \tau_{3j_m+p_m}| = |\theta_{3j_m+p_m} - \tau_{3j_1+p_1}| \leq \varepsilon \quad \text{for } m = 1, \dots, s-1 \quad (4.17)$$

and

$$\tau_{3j_m+p_m} \geq \tau_{3j_s+p_s} > \tau_{3j_1+p_1} + \sqrt{\varepsilon} \quad \text{for } m = s, \dots, N - N_*.$$

The lower bound (4.13) and inequality (4.15) imply

$$x_{3j_m+p_m}(\tau_{3j_1+p_1} - \varepsilon) \geq C\sqrt{\varepsilon}^{1/(\beta+1)} \quad \text{and} \quad y_{3j_m+p_m}(\tau_{3j_1+p_1} - \varepsilon) \geq C\sqrt{\varepsilon}^{1/(\beta+1)}$$

for $m = s, \dots, N - N_*$ and all sufficiently small δ . Using (4.12) for $y_{3j_m+p_m}$ we find

$$\theta_{3j_m+p_m} \geq \tau_{3j_1+p_1} + C\sqrt{\varepsilon} \geq \tau_{3j_1+p_1} + \varepsilon,$$

and (4.13) gives

$$y_{3j_m+p_m}(t) \geq C(\theta_{3j_m+p_m} - t)^{1/(\beta+1)} \geq C\varepsilon^{1/(\beta+1)}$$

for all $t \in [\tau_{3j_1+p_1} - \varepsilon, \tau_{3j_1+p_1} + \varepsilon]$ and $m = s, \dots, N - N_*$. Consequently, we have

$$\int_{\tau_{3j_1+p_1}-\varepsilon}^{\tau_{3j_1+p_1}+\varepsilon} y_{3j_m+p_m}(t)^{-\beta} dt \leq C\varepsilon^{-\beta/(\beta+1)}\varepsilon = C\varepsilon^{1-\frac{\beta}{\beta+1}}$$

for $m = s, \dots, N - N_*$, while for $m = 1, \dots, s - 1$ the bounds (4.17) and (4.12) imply

$$\int_{\tau_{3j_1+p_1}-\varepsilon}^{\tau_{3j_1+p_1}+\varepsilon} y_{3j_m+p_m}(t)^{-\beta} dt \leq \int_{\tau_{3j_1+p_1}-\varepsilon}^{\theta_{3j_m+p_m}} C(\theta_{3j_m+p_m} - t)^{-\frac{\beta}{\beta+1}} dt \leq C\varepsilon^{1-\frac{\beta}{\beta+1}}.$$

Furthermore, the large particles $k < 0$ and $k = 3j$, $j = 0, \dots, N$ clearly satisfy

$$\int_{\tau_{3j_1+p_1}-\varepsilon}^{\tau_{3j_1+p_1}+\varepsilon} y_k(t)^{-\beta} dt \leq C\varepsilon \leq C\varepsilon^{1-\frac{\beta}{\beta+1}}.$$

Using these inequalities in the equations for all particles y_k that have not vanished at time $\tau_{3j_1+p_1} + \varepsilon$, that is, for all $k \notin \{k = 3j_m + p_m : m = 1, \dots, s - 1\}$, we obtain

$$|y_k(\tau_{3j_1+p_1} + \varepsilon) - y_k(\tau_{3j_1+p_1} - \varepsilon)| \leq C\varepsilon^{1-\frac{\beta}{\beta+1}},$$

and since a similar computation applies to each x_k , $k \notin \{k = 3j_m + p_m : m = 1, \dots, s - 1\}$ we deduce

$$|D_k(\tau_{j_1+p_1} + \varepsilon)| \leq |D_k(\tau_{j_1+p_1} - \varepsilon)| + C\varepsilon^{1-\frac{\beta}{\beta+1}} \leq \omega + C\varepsilon^{1-\frac{\beta}{\beta+1}}.$$

Thus, at time $\tau_{j_1+p_1} + \varepsilon$ we have $x_{3j_m+p_m} = y_{3j_m+p_m} = 0$ for $m = 1, \dots, s - 1$, while the other particles are positive and arbitrarily close to each other if δ is sufficiently small. Repeating the preceding arguments we obtain the asserted continuity after at most $2N$ steps. \square

To study the vanishing times as particle sizes become small we consider the local problem

$$\begin{aligned} \dot{Y}_1 &= -2Y_1^{-\beta} + Y_2^{-\beta} + F_1, \\ \dot{Y}_2 &= -2Y_2^{-\beta} + Y_1^{-\beta} + F_2, \end{aligned} \tag{4.18}$$

where F_1 and F_2 are continuous functions of time. Solutions to (4.18) are understood in the sense that vanished particles are removed, and their existence for positive initial data follows as in Section 3.

Lemma 4.4 (Vanishing times for local equation). *Let (Y_1, Y_2) be a solution to (4.18) with initial data $Y_1(0) = A_1$ and $Y_2(0) = A_2$, where $1/2 \leq A_p \leq 2$ and $|F_1| + |F_2| \leq \eta$. Then the following properties hold:*

1. *For all sufficiently small η depending only on β both vanishing times, denoted by τ_1 and τ_2 , are finite independently of A_p and F_p . Moreover, $Y_1 + Y_2 \leq 4$ for all t .*
2. *Given $\varepsilon > 0$ there exists $\eta_0 = \eta_0(\beta, \varepsilon) \sim \varepsilon$ such that for all $\eta < \eta_0$ the conditions $A_1 - A_2 = \varepsilon > 0$ and $|F_1| + |F_2| \leq \eta$ imply $\tau_2 < \tau_1$ and $Y_1(t) - Y_2(t) \geq \varepsilon/2$ for all $t \in [0, \tau_2]$.*
3. *With $\nu = |A_1 - A_2|$ and η as above we have*

$$\max_{p=1,2} \left| \tau_p - \frac{A_p^{\beta+1}}{\beta+1} \right| \rightarrow 0 \quad \text{as} \quad (\nu, \eta) \rightarrow (0, 0)$$

independently of F_p and A_p , $p = 1, 2$.

Proof. Until the first vanishing time we have

$$\dot{Y}_1 + \dot{Y}_2 = -(Y_1^{-\beta} + Y_2^{-\beta}) + F_1 + F_2 \leq -(Y_1 + Y_2)^{-\beta} + \eta,$$

and since $A_1 + A_2 \leq 4$ the sum $Y_1 + Y_2$ decreases for all $\eta < 4^{-\beta}/2$. In fact, since $\eta < (A_1 + A_2)^{-\beta}/2$, we have

$$\dot{Y}_1 + \dot{Y}_2 \leq -\frac{1}{2}(Y_1 + Y_2)^{-\beta},$$

thus the first particle vanishes at a time smaller than $C(A_1 + A_2)^{\beta+1} \leq 4^{\beta+1}C$. Thereafter, the same argument applied to the remaining particle shows that the second vanishing time is bounded by $C(A_1 + A_2)^{\beta+1} \leq 4^{\beta+1}C$, too.

As long $Y_1 > Y_2 > 0$, which by the assumption for the second claim is true initially, the difference $Y_1 - Y_2$ satisfies

$$\dot{Y}_1 - \dot{Y}_2 = -3(Y_1^{-\beta} - Y_2^{-\beta}) + F_1 - F_2 \geq -\eta,$$

and we obtain

$$Y_1(t) - Y_2(t) \geq \varepsilon - \eta t \geq \varepsilon - \eta \max(\tau_1, \tau_2).$$

Hence, the second assertion follows with $\eta_0 = \varepsilon/(2 \max(\tau_1, \tau_2))$.

To prove the third claim, set $Y_{\max}(t) = \max(Y_1(t), Y_2(t))$ and $Y_{\min}(t) = \min(Y_1(t), Y_2(t))$ and denote the corresponding vanishing times by τ_{\max} and τ_{\min} , respectively. Letting $0 < \omega < 2^{-(\beta+1)}$ be arbitrary and setting $\tau = (1 - \omega)Y_{\max}(0)^{\beta+1}/(\beta + 1)$ we are going to show that $\tau \leq \tau_{\min} \leq \tau_{\max} \leq \tau + O(\omega)$ as $(\nu, \eta) \rightarrow 0$, which yields the result due to $|Y_{\max}(0) - A_p| \leq |A_1 - A_2| = \nu$ for $p = 1, 2$.

First, since $Y_{\max} \leq 4$ we have $\eta Y_{\max}^{\beta} \leq \omega$ for all $t > 0$ and all $\eta < 4^{-\beta}\omega$. Using $Y_{\max} \geq Y_{\min}$ we thus find

$$\dot{Y}_{\max}(t) \geq -Y_{\max}(t)^{-\beta} - \eta \geq -(1 + \omega)Y_{\max}(t)^{-\beta}$$

for almost all $t < \tau_{\min}$ and conclude

$$Y_{\max}(t) \geq \left(Y_{\max}(0)^{\beta+1} - (\beta + 1)(1 + \omega)t \right)^{\frac{1}{\beta+1}} \geq Y_{\max}(0)\omega^{\frac{2}{\beta+1}} \quad (4.19)$$

if $t \leq \min(\tau, \tau_{\min})$. Next, as $Y_{\min}(0) = Y_{\max}(0) - \nu$ we have

$$Y_{\min}(t) \geq \frac{1}{2}(Y_{\max}(t) - \nu) \geq \frac{1}{2}\left(Y_{\max}(0)\omega^{\frac{2}{\beta+1}} - \nu \right) \geq \frac{1}{4}Y_{\max}(0)\omega^{\frac{2}{\beta+1}} \quad (4.20)$$

for small $t > 0$ and all $\nu \leq Y_{\max}(0)\omega^{2/(\beta+1)}/2$. For such t we find

$$\begin{aligned} \dot{Y}_{\max} - \dot{Y}_{\min} &\leq 3(Y_{\min}^{-\beta} - Y_{\max}^{-\beta}) + \eta = 3\beta \int_{Y_{\min}}^{Y_{\max}} y^{-(\beta+1)} dy + \eta \\ &\leq C \frac{1}{Y_{\max}(0)^{\beta+1}\omega^2} (Y_{\max} - Y_{\min}) + \eta, \end{aligned}$$

and choosing ν and η sufficiently small we infer that

$$Y_{\max}(t) - Y_{\min}(t) \leq C(\nu + \eta\omega^2) e^{C \frac{\min(\tau, \tau_{\min})}{Y_{\max}(0)^{\beta+1}\omega^2}} + C\eta\omega^2 \leq Y_{\max}(0)\omega^{\frac{3}{\beta+1}}. \quad (4.21)$$

Combining (4.19) and (4.21) we arrive at

$$Y_{\min}(t) = Y_{\max}(t) + Y_{\min}(t) - Y_{\max}(t) \geq Y_{\max}(0)\omega^{\frac{2}{\beta+1}}(1 - \omega^{\frac{1}{\beta+1}}) \geq \frac{1}{2}Y_{\max}(0)\omega^{\frac{2}{\beta+1}}, \quad (4.22)$$

which implies that by continuation the above considerations extend to $\min(\tau, \tau_{\min})$. Therefore, $\tau < \tau_{\min}$ and we have

$$\frac{Y_{\max}(0)^{\beta+1}}{\beta + 1} - O(\omega) \leq \tau_{\min} \leq \tau_{\max}.$$

Moreover, we find

$$\begin{aligned} Y_{\min}^{-\beta} - Y_{\max}^{-\beta} &= \beta \int_{Y_{\min}}^{Y_{\max}} y^{-(\beta+1)} dy \leq \beta Y_{\min}^{-(\beta+1)} (Y_{\max} - Y_{\min}) \\ &\leq \beta \left(\frac{Y_{\max}}{Y_{\min}} \right)^{\beta+1} \frac{Y_{\max} - Y_{\min}}{Y_{\max}} Y_{\max}^{-\beta} \leq C \omega^{\frac{1}{\beta+1}} Y_{\max}^{-\beta} \end{aligned}$$

using (4.21)–(4.22) to estimate the first fraction and (4.20)–(4.21) for the second. From this and with $\eta Y_{\max}^{\beta} \leq \omega$ in the equation for Y_{\max} we deduce

$$\dot{Y}_{\max}(t) \leq -Y_{\max}(t)^{-\beta} + C \omega^{\frac{1}{\beta+1}} Y_{\max}(t)^{-\beta} + \eta \leq -\left(1 - C \omega^{\frac{1}{\beta+1}} - \omega\right) Y_{\max}(t)^{-\beta}$$

for $t < \tau$. By integration we obtain

$$Y_{\max}(\tau)^{\beta+1} \leq Y_{\max}(0)^{\beta+1} - (\beta+1)(1 - C \omega^{\frac{1}{\beta+1}} - \omega)\tau = Y_{\max}(0)^{\beta+1} r(\omega)$$

where $r(\omega) \rightarrow 0$ as $\omega \rightarrow 0$. The argument that proved the first part of the lemma but with η depending on ω , now yields $\tau_{\min} \leq \tau_{\max} \leq \tau + C Y_{\max}(0)^{\beta+1} r(\omega)^{\beta+1}$. \square

Finally, we prove that the sequences $(R_{j,p}^N)$ can be chosen such that neighboring small particles of the solution \bar{x} for $\varepsilon^N = 0$ vanish at the same time.

Proposition 4.5. *There exists $N_* > 0$ such that for all $N > N_*$ and any sequence (R_j) as in (4.9) there are two sequences $(R_{j,p}^N)_{j=0,\dots,N-1}$, $p = 1, 2$ satisfying (4.10) with the following properties.*

1. *The vanishing times $\bar{\tau}_{3j+p}^N$, $p = 1, 2$ are identical for $j = N_*, \dots, N-1$.*
2. *We have $R_{j,1}^N = R_j$ for $j = 0, \dots, N-1$ and $R_{j,2}^N = R_j$ for $j = 0, \dots, N_*-1$ as well as*

$$|R_{j,2}^N - R_j| \leq \omega_j R_j \quad \text{for } j = N_*, \dots, N-1,$$

where (ω_j) is a sequence with $\lim_{j \rightarrow \infty} \omega_j = 0$ that is independent of (R_j) .

3. *There is a decreasing sequence $(\tilde{\omega}_j)$ independent of (R_j) with $0 < \tilde{\omega}_j < 1/(4(\beta+1))$ and $\lim_{j \rightarrow \infty} \tilde{\omega}_j = 0$ such that*

$$\left| \bar{\tau}_{3j+p}^N - \frac{R_j^{\beta+1}}{\beta+1} \right| \leq \tilde{\omega}_j R_j^{\beta+1}$$

for $j = N_, \dots, N-1$ and $p = 1, 2$.*

Proof. For N_* larger than in Lemma 4.3 to be fixed later and $N > N_*$ we define a map Φ from the cube $Q = [-1, 1]^{N-N_*}$ to \mathbb{R}^{N-N_*} as follows. Given $\eta = (\eta_k)_{k=N_*}^{N-1} \in Q$ we solve (2.2) for \bar{x}^N with $R_{j,1}^N = R_j$ for $j = 0, \dots, N-1$, $R_{j,2}^N = R_j$ for $j = 0, \dots, N_*-1$, and $R_{j,2}^N = R_j(1 + \eta_j \omega_j)$ for $j = N_*, \dots, N-1$, where (R_j) is as in (4.9) and $(\omega_j)_{j \geq 0}$ a sequence of positive numbers also to be fixed later. Then we set

$$\Phi(\eta) = (\bar{\tau}_{3j+1}^N - \bar{\tau}_{3j+2}^N)_{j=N_*}^{N-1}.$$

By Lemma 4.3 the function Φ is well-defined, finite and continuous. We aim to show that the topological degree $\deg(\Phi, Q, 0)$ is nonzero, which implies the existence of $\eta_* \in Q$ such that $\Phi(\eta_*) = 0$, that is, $\bar{\tau}_{3j+1}^N = \bar{\tau}_{3j+2}^N$ for $j = N_*, \dots, N-1$. For that purpose, we use that the degree is invariant under the homotopy

$$\Psi(\lambda, \eta) = -(1 - \lambda)\eta + \lambda\Phi(\eta), \quad \lambda \in [0, 1], \quad \eta \in Q,$$

and thus

$$1 = \deg(-\text{Id}, Q, 0) = \deg(\Psi(0, \cdot), Q, 0) = \deg(\Psi(1, \cdot), Q, 0) = \deg(\Phi, Q, 0),$$

provided that $0 \neq \Psi(\lambda, \eta)$ for all $\lambda \in [0, 1]$ and $\eta \in \partial Q$. To compute $\Psi(\lambda, \eta)$ suppose first that $\eta \in \partial Q$ satisfies $\eta_m = -1$ for some $m \in \{N_*, \dots, N-1\}$ and $|\eta_k| \leq 1$ otherwise. Setting

$$y_1(t) = \frac{\bar{x}_{3m+1}^N(R_j^{\beta+1}t)}{R_m}, \quad y_2(t) = \frac{\bar{x}_{3m+2}^N(R_j^{\beta+1}t)}{R_m}$$

and $A_1 = 1$, $A_2 = 1 - \omega_m$ we aim to apply Lemma 4.4 with

$$F_1(t) = \left(\frac{\bar{x}_{3m}^N(R_m^{\beta+1}t)}{R_m} \right)^{-\beta}, \quad F_2(t) = \left(\frac{\bar{x}_{3m+3}^N(R_m^{\beta+1}t)}{R_m} \right)^{-\beta}.$$

Since $A_1 - A_2 = \omega_m$ and $|F_1| + |F_2| \leq CR_m^\beta$ by Proposition 4.2, we need $CR_m^\beta \leq \eta_0(\beta, \omega_m) \sim \omega_m$, which due to $R_j = \gamma^j$ can be arranged for some decreasing sequence (ω_j) independent of $R_j = \gamma^j$ and $\gamma < 1/3$. We thus find $\bar{\tau}_{3m+1}^N > \bar{\tau}_{3m+2}^N$, that is, $\Phi(\eta)_m > 0$ and $\Psi(\lambda, \eta)_m = (1 - \lambda) + \Phi(\eta)_m > 0$ for $\lambda \in [0, 1]$. Finally, the same argument yields $\Psi(\lambda, \eta)_m < 0$ if $\eta_m = +1$.

The sequences $(R_{j,p}^N)$ corresponding to $\eta_* \in Q$, whose existence is now proven, satisfy the second claim of the proposition by definition of Φ . The last assertion is a consequence of Lemma 4.4(3) applied to $y_p(t) = x_{3j+p}^N(R_j^{\beta+1}t)/R_j$ for $j = N_*, \dots, N-1$ and $p = 1, 2$, which yields

$$\left| \frac{\bar{\tau}_{3j+p}^N}{R_j^{\beta+1}} - \frac{1}{\beta+1} \left(\frac{R_{j,p}^N}{R_j} \right)^{\beta+1} \right| \rightarrow 0 \quad \text{as} \quad \omega_j \rightarrow 0,$$

and finishes the proof. \square

Remark. Exactly as in the proof above, we can apply Lemma 4.4 also to x and obtain

$$\left| \tau_{3j+p}^N - \frac{R_j^{\beta+1}}{\beta+1} \right| \leq \tilde{\omega}_j R_j^{\beta+1}$$

for $j = N_*, \dots, N-1$ and $p = 1, 2$, where N_* and $\tilde{\omega}_j$ are as in Proposition 4.5.

4.3 Auxiliary results for the local problem

In this section, we collect some auxiliary results for the local problem (4.18), which we use to study the propagation of the perturbation. The first two lemmas address the asymptotic behavior of solutions.

Lemma 4.6 (Power law for simultaneously vanishing particles). *Given $F_1, F_2 \in C([0, \infty))$, suppose that $Y_1, Y_2 \in C^1(0, \bar{\tau}) \cap C^0([0, \bar{\tau}])$ solve (4.18) with initial data $Y_1(0), Y_2(0) \in [R/2, 3R/2]$ for some $R > 0$ and vanish simultaneously at time $\bar{\tau} > 0$. Then there is a constant C such that with $\eta = \|F_1\|_\infty + \|F_2\|_\infty$ we have*

$$\left| Y_p(t) - ((\beta+1)(\bar{\tau}-t))^{\frac{1}{\beta+1}} \right| \leq C\eta(\bar{\tau}-t) \leq CR^{\beta+1}$$

for all $t \in [0, \bar{\tau}]$ and $p = 1, 2$.

Proof. The arguments for estimate (3.2) and Lemma 3.5 (with $L = 3$ and $d = 1/2$) apply to Y_1, Y_2 and yield

$$0 < c \leq \frac{Y_p(t)}{(\bar{\tau} - t)^{\frac{1}{\beta+1}}} \leq C < \infty \quad (4.23)$$

for $p = 1, 2$ and all $t \in [0, \bar{\tau})$. We define a new time $s \in [s_0, \infty)$ by

$$R^{\beta+1}e^{-s(\beta+1)} = \bar{\tau} - t, \quad R^{\beta+1}e^{-s_0(\beta+1)} = \bar{\tau}$$

and set

$$W_p(s) = \frac{Y_p(t)}{(\bar{\tau} - t)^{\frac{1}{\beta+1}}} = \frac{1}{R}e^s Y_p(t), \quad p = 1, 2.$$

Then (4.23) implies that $W_p(s) \in [c, C]$ for all $s \geq s_0$ and that s_0 is bounded from above and below independently of R , because $\bar{\tau}$ is of order $R^{\beta+1}$. Moreover, W_p satisfies

$$\frac{d}{ds}W_p(s) = W_p(s) + (\beta + 1) \left(-2W_p(s)^{-\beta} + W_{3-p}(s)^{-\beta} \right) + (\beta + 1)R^\beta e^{-s\beta} F_p(t)$$

for $p = 1, 2$, and taking the difference of both equations we obtain

$$\frac{d}{ds}(W_1 - W_2) = W_1 - W_2 - 3(\beta + 1) \left(W_1^{-\beta} - W_2^{-\beta} \right) + (\beta + 1)R^\beta e^{-s\beta} (F_1 - F_2).$$

Next, multiplying by $\text{sign}(W_1 - W_2)$ and observing that $(W_1^{-\beta} - W_2^{-\beta}) \text{sign}(W_1 - W_2) \leq 0$ we find

$$\frac{d}{ds}|W_1 - W_2| \geq |W_1 - W_2| - (\beta + 1)R^\beta e^{-s\beta} \eta$$

where we also used $\|F_1\|_\infty + \|F_2\|_\infty \leq \eta$. Integration then yields

$$\begin{aligned} |W_1 - W_2|(s_2) &\geq \left(|W_1 - W_2|(s_1) - (\beta + 1)R^\beta \eta \int_{s_1}^{s_2} e^{-s\beta} e^{-(s-s_1)} ds \right) e^{s_2-s_1} \\ &\geq \left(|W_1 - W_2|(s_1) - (\beta + 1)R^\beta \eta e^{-s_1\beta} \right) e^{s_2-s_1} \end{aligned} \quad (4.24)$$

for all $s_2 \geq s_1 \geq s_0$. We conclude that

$$|W_1 - W_2|(s) \leq (\beta + 1)R^\beta \eta e^{-s\beta} \quad \text{for all } s \geq s_0, \quad (4.25)$$

because otherwise the existence of $s_1 \geq s_0$ such that $|W_1 - W_2|(s_1) \geq (1+\nu)(\beta+1)R^\beta \eta e^{-s_1\beta}$ for some $\nu > 0$ and (4.24) imply that $|W_1 - W_2|(s)$ grows exponentially for $s \geq s_1$ in contradiction to $W_p \in [c, C]$.

We now use (4.25) and $W_p \geq c$ to linearize and estimate

$$|W_1^{-\beta} - W_2^{-\beta}|(s) \leq C|W_1 - W_2|(s) \leq CR^\beta \eta e^{-s\beta}$$

for $p = 1, 2$, and we deduce that

$$\frac{d}{ds}W_p(s) = W_p(s) - (\beta + 1)W_p(s)^{-\beta} + R^\beta \eta e^{-s\beta} \tilde{F}_p(s)$$

where \tilde{F}_p is bounded by a constant depending only on β . Standard ODE methods imply that the only solution of $\dot{W} = W - (\beta + 1)W^{-\beta}$ with $0 < c \leq W \leq C$ is the constant function $W_0 \equiv (\beta + 1)^{1/(\beta+1)}$, and arguing by contradiction as above we obtain

$$|W_p(s) - W_0| \leq C\|\tilde{F}_p\|_\infty \eta R^\beta e^{-s\beta}$$

for all $s \geq s_0$ or, equivalently,

$$\left| \frac{Y_p(t)}{(\bar{\tau} - t)^{\frac{1}{\beta+1}}} - (\beta + 1)^{\frac{1}{\beta+1}} \right| \leq C\|\tilde{F}_p\|_\infty \eta (\bar{\tau} - t)^{\frac{\beta}{\beta+1}}$$

for all $t \in [0, \bar{\tau}]$ and $p = 1, 2$. □

Lemma 4.7 (Asymptotics near vanishing). *Suppose that $\beta \geq \beta_*$ where β_* is as in (4.2). There are constants $B_* > 0$, $S_* \in (-1, 1)$, $\varepsilon_0 > 0$, $\eta_0 > 0$, $T_0 > 0$ such that the following result is true: If Y_1, Y_2 solve (4.18) (with our usual convention that $0^{-\beta} = 0$ and that Y_p remains 0 once it has vanished) in the interval $(-T, \infty)$, where F_p are continuous such that $\|F\|_\infty \leq \eta \leq \eta_0$, $T > T_0$ and $\eta^{1-\varepsilon_0} \leq T^{-(4\beta+1)/(\beta+1)}$, and if*

$$\left| \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} (-T) - (\beta+1)^{\frac{1}{\beta+1}} T^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - B_* T^{-\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right| \leq C (\eta^{\varepsilon_0} + T^{-1}) T^{-\frac{3\beta}{\beta+1}},$$

then the vanishing times τ_p of Y_p satisfy

$$|\tau_1 - (S_* + 1)| + |\tau_2 - (S_* - 1)| = o(1)_{\eta \rightarrow 0} + o(1)_{T \rightarrow \infty}.$$

Proof. The proof consists of four steps. In the first three steps we construct a solution for (4.18) and $F_p = 0$ which has the desired asymptotic behavior as $t \rightarrow -\infty$. In the final step we show that we reach every solution with sufficiently small $\|F_p\|$.

Step 1: Backwards solution. The equation $\dot{Y}_1 = -2Y_1^{-\beta}$ in $(-1, 1)$ with terminal data $Y_1(1) = 0$ has the unique solution

$$Y_1(t) = (2(\beta+1)(1-t))^{\frac{1}{\beta+1}},$$

and we consider

$$\dot{Y}_1 = -2Y_1^{-\beta} + Y_2^{-\beta}, \quad \dot{Y}_2 = -2Y_2^{-\beta} + Y_1^{-\beta}$$

in $(-\infty, -1)$ with data $Y_1(-1) = (4(\beta+1))^{1/(\beta+1)}$ and $Y_2(-1) = 0$. A local solution which is positive for $t < -1$ exists, and since $Y_{\min} = \min(Y_1, Y_2)$ and $Y_{\max} = \max(Y_1, Y_2)$ satisfy

$$\dot{Y}_{\min} \leq -Y_{\min}^{-\beta} \quad \text{and} \quad \dot{Y}_{\max} \geq -Y_{\max}^{-\beta}$$

we obtain

$$((\beta+1)(-1-t))^{\frac{1}{\beta+1}} \leq Y_{\min}(t) \leq Y_{\max}(t) \leq ((\beta+1)(3-t))^{\frac{1}{\beta+1}}.$$

Thus, global existence follows and we find $Y_1 > Y_2$.

In order to study the asymptotic behavior of $Y_p(t)$ as $-t$ becomes large, we let

$$W_p(s) = \frac{Y_p(t)}{(-t)^{\frac{1}{\beta+1}}} = Y_p(t) e^{-\frac{s}{\beta+1}} \quad \text{where} \quad -t = e^s, \quad s \geq 0,$$

which satisfies

$$\frac{d}{ds} W_p = -\frac{W_p}{\beta+1} + 2W_p^{-\beta} - W_{3-p}^{-\beta}. \quad (4.26)$$

Subtracting the equations for W_1 and W_2 from each other and multiplying by $\text{sign}(W_1 - W_2)$ we deduce

$$\frac{d}{ds} |W_1 - W_2| \leq -\frac{|W_1 - W_2|}{\beta+1}$$

and

$$|W_1(s) - W_2(s)| \leq |W_1(0) - W_2(0)| e^{-\frac{s}{\beta+1}} = (4(\beta+1))^{\frac{1}{\beta+1}} e^{-\frac{s}{\beta+1}}$$

for all $s \in (0, \infty)$. With this estimate and the lower bound

$$W_p(s) \geq Y_{\min}(t) e^{-\frac{s}{\beta+1}} \geq C(-1 + e^s)^{\frac{1}{\beta+1}} e^{-\frac{s}{\beta+1}} = C(1 - e^{-s})^{\frac{1}{\beta+1}}$$

we linearize $W_1^{-\beta} - W_2^{-\beta}$ to get

$$|W_1^{-\beta} - W_2^{-\beta}| \leq \beta W_{\min}^{-(\beta+1)} |W_1 - W_2| \leq C \frac{1}{1 - e^{-s}} e^{-\frac{s}{\beta+1}}$$

and

$$\frac{d}{ds}W_p = -\frac{W_p}{\beta+1} + W_p^{-\beta} + O\left(\frac{e^{-\frac{s}{\beta+1}}}{1-e^{-s}}\right)$$

for $s > 0$. As in the proof of Lemma 4.6 it follows that $w_p = W_p - (\beta+1)^{\frac{1}{\beta+1}}$ satisfies

$$|w_p(s)| \leq C \frac{e^{-\frac{s}{\beta+1}}}{1-e^{-s}}$$

for $s > 0$, and since $w_p(0) = Y_p(-1) - (\beta+1)^{1/(\beta+1)}$ we conclude that $|w_p(s)| \leq Ce^{-s/(\beta+1)}$ for all $s \geq 0$. Thus, linearizing (4.26) around $(\beta+1)^{1/(\beta+1)}$ we obtain

$$\frac{d}{ds} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{-1}{\beta+1} \begin{pmatrix} 2\beta+1 & -\beta \\ -\beta & 2\beta+1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + O(w_1^2 + w_2^2). \quad (4.27)$$

for s sufficiently large, say $s > s_0$. The solution of (4.27) can be written as

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(s) = \Psi(s) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(s_0) + \int_{s_0}^s \Psi(s-r) O(w_1(r)^2 + w_2(r)^2) dr,$$

where

$$\Psi(s) = \frac{1}{2}e^{-(s-s_0)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}e^{-(s-s_0)\frac{3\beta+1}{\beta+1}} \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$$

is the fundamental matrix for the linear part of (4.27). The contributions from $w_1^2 + w_2^2$ are of order

$$\int_{s_0}^s e^{-(s-r)} e^{-2r} dr = e^{-s} \int_{s_0}^{\infty} e^{-r} dr - e^{-s} \int_s^{\infty} e^{-r} dr = e^{-s_0} e^{-s} + O(e^{-2s})$$

and

$$\int_0^s e^{-(s-r)\frac{3\beta+1}{\beta+1}} e^{-2r\frac{3\beta+1}{\beta+1}} dr = \frac{\beta+1}{3\beta+1} e^{-s_0\frac{3\beta+1}{\beta+1}} e^{-s\frac{3\beta+1}{\beta+1}} + O(e^{-2s\frac{3\beta+1}{\beta+1}})$$

with no mixed terms appearing since the eigenvectors corresponding to both powers in Ψ are orthogonal. As a consequence we have

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = A_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-s} + B_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-s\frac{3\beta+1}{\beta+1}} + O(e^{-2s} + e^{-2s\frac{3\beta+1}{\beta+1}}) \quad (4.28)$$

for all $s \geq s_0$, where A_0, B_0 are two constants and the contribution of order e^{-2s} is present in w_p if and only if the contribution of order e^{-s} is.

Step 2: Removing the order e^{-s} . For a given shift $S_* \in \mathbb{R}$ let $\tilde{Y}_p(t) = Y_p(t - S_*)$ and set

$$\tilde{Y}_p(t) e^{-\frac{s}{\beta+1}} = \tilde{W}_p(s) = (\beta+1)^{\frac{1}{\beta+1}} + \tilde{w}_p(s)$$

for $t < \min(0, S_* - 1)$ and $e^s = -t$. Repeating the considerations from Step 1 we find

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = A_{S_*} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-s} + B_{S_*} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-s\frac{3\beta+1}{\beta+1}} + O(e^{-2s} + e^{-2s\frac{3\beta+1}{\beta+1}}) \quad (4.29)$$

for some constants A_{S_*}, B_{S_*} and $s \geq s_{S_*}$ where the s_{S_*} is chosen appropriately; as above, the contribution of order e^{-2s} is present if and only if the term of order e^{-s} is. On the

other hand, writing $-t + S_* = e^s + S_* = e^s(1 + S_*e^{-s}) = e^{\tilde{s}}$ and linearizing $(1 + S_*e^{-s})^\alpha = 1 + \alpha S_*e^{-s} + O(e^{-2s})$ we deduce

$$\begin{aligned}\widetilde{W}_p(s) &= \widetilde{Y}_p(t)e^{-\frac{s}{\beta+1}} = Y_p(t - S_*)e^{-\frac{\tilde{s}}{\beta+1}}e^{\frac{\tilde{s}-s}{\beta+1}} = W_p(\tilde{s})(1 + S_*e^{-s})^{\frac{1}{\beta+1}} \\ &= W_p(\tilde{s}) \left(1 + \frac{1}{\beta+1} S_*e^{-s} + O(e^{-2s}) \right)\end{aligned}$$

and thus

$$\widetilde{w}_p(s) = (\beta+1)^{-\frac{\beta}{\beta+1}} S_*e^{-s} + w_p(\tilde{s}) + \frac{1}{\beta+1} S_*e^{-s} w_p(\tilde{s}) + O(e^{-2s}).$$

Formula (4.28) for w_p and the definition of \tilde{s} imply

$$\begin{aligned}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(\tilde{s}) &= \frac{A_0 e^{-s}}{1 + S_*e^{-s}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{B_0 e^{-s \frac{3\beta+1}{\beta+1}}}{(1 + S_*e^{-s})^{\frac{3\beta+1}{\beta+1}}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O\left(e^{-2s} + e^{-2s \frac{3\beta+1}{\beta+1}}\right) \\ &= w_p(s) + O\left(e^{-2s} + e^{-s \left(\frac{3\beta+1}{\beta+1} + 1\right)}\right),\end{aligned}$$

so that by combining the previous two equations we obtain

$$\begin{aligned}\begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{pmatrix} &= \left((\beta+1)^{-\frac{\beta}{\beta+1}} S_* + A_0 \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-s} + B_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-s \frac{3\beta+1}{\beta+1}} \\ &\quad + O\left(e^{-2s} + e^{-3s} + e^{-s \left(\frac{3\beta+1}{\beta+1} + 1\right)}\right)\end{aligned}\quad (4.30)$$

as a second representation for \widetilde{w} . With $S_* = -(\beta+1)^{\beta/(\beta+1)} A_0$ and $B_* = B_0$ the term of order e^{-s} in (4.30) vanishes and thus (4.29) implies that there is no contribution of order e^{-2s} , either. Comparing the orders of the remaining error terms we arrive at

$$\begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{pmatrix} = B_* \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-s \frac{3\beta+1}{\beta+1}} + O\left(e^{-s \left(\frac{3\beta+1}{\beta+1} + 1\right)}\right)$$

for $s \geq s_{S_*}$ or, equivalently,

$$\begin{pmatrix} \widetilde{Y}_1 \\ \widetilde{Y}_2 \end{pmatrix} = (\beta+1)^{\frac{1}{\beta+1}} (-t)^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B_* (-t)^{-\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O\left((-t)^{-\frac{3\beta}{\beta+1}-1}\right)\quad (4.31)$$

for $t \leq -e^{s_{S_*}}$.

Step 3: Properties of B_ and S_* .* Since $\widetilde{Y}_1 > \widetilde{Y}_2$, we obviously have $B_* \geq 0$. Moreover, if $B_* = 0$ then (4.27) implies $\widetilde{w} \equiv 0$, which contradicts the construction of \widetilde{Y} . Thus, we have $B_* > 0$.

Next, we show $S_* < 1$ by comparing \widetilde{Y} with $\bar{Y}(t) = [(\beta+1)(-t)]^{1/(\beta+1)}$. Suppose for contradiction that $S_* \geq 1$ so that $\widetilde{Y}_1(0) > 0$ and $\widetilde{Y}_2(0) \geq 0$. Then we have $\widetilde{Y}_1(t) > \bar{Y}(t)$ for small $(-t)$, but also $\widetilde{Y}_2(t) > \bar{Y}(t)$, because either $\widetilde{Y}_2(0) > 0$ or

$$\frac{d}{dt} \widetilde{Y}_2 = -2\widetilde{Y}_2^{-\beta} + \widetilde{Y}_1^{-\beta} < -\frac{3}{2}\widetilde{Y}_2^{-\beta}$$

for small $(-t)$ such that $\widetilde{Y}_2^\beta < \widetilde{Y}_1^\beta/2$. Assume now that there is a first time $t_0 < 0$ such that $\widetilde{Y}_1(t_0) = \bar{Y}(t_0)$ or $\widetilde{Y}_2(t_0) = \bar{Y}(t_0)$. Then clearly $\widetilde{Y}_1(t_0) \neq \widetilde{Y}_2(t_0)$, because otherwise uniqueness for the ODE yields $\widetilde{Y}_1(t) = \widetilde{Y}_2(t) = \bar{Y}(t)$ for all $t < 0$ in contradiction to the asymptotics of \widetilde{Y}_p proved above. However, if $\widetilde{Y}_2(t_0) = \bar{Y}(t_0) < \widetilde{Y}_1(t_0)$ then we find

$$0 \leq \frac{d}{dt}(\widetilde{Y}_2 - \bar{Y})(t_0) = \widetilde{Y}_1(t_0)^{-\beta} - \bar{Y}(t_0)^{-\beta} < 0,$$

and in the same way the case $\tilde{Y}_1(t_0) = \bar{Y}(t_0) < \tilde{Y}_2(t_0)$ is excluded. Thus, we conclude that $\tilde{Y}_1(t) > \bar{Y}(t)$ and $\tilde{Y}_2(t) > \bar{Y}(t)$ for all $t < 0$, which again contradicts the asymptotics of \tilde{Y} .

To prove $S_* > -1$, we consider the phase portrait of (Z_1, Z_2) where $Z_p(s) = \tilde{W}_p((\beta + 1)s)/(\beta + 1)^{1/(\beta+1)}$, $p = 1, 2$. From the previous steps we know that

$$\begin{aligned} \frac{d}{ds} Z_1 &= -Z_1 + 2Z_1^{-\beta} - Z_2^{-\beta}, \\ \frac{d}{ds} Z_2 &= -Z_2 + 2Z_2^{-\beta} - Z_1^{-\beta}, \end{aligned} \quad (4.32)$$

and that the trajectory $(Z_1, Z_2)(s)$ approaches the point $(1, 1)$ parallel to the vector $(-1, 1)$ as $s \rightarrow \infty$. On the other hand, if $S_* \leq 1$ we have $Z_2(s_0) = 0$ and $0 < Z_1(s_0) \leq 2^{1/(\beta+1)}$ where s_0 is the time corresponding to $t = S_* - 1$, and a calculation shows that on the line which connects $(1, 1)$ to $(2^{1/(\beta+1)}, 0)$ the dynamics of (4.32) point to the left of that line if $\beta \geq \beta_*$; compare Figure 4.1. As this contradicts the asymptotics of (Z_1, Z_2) , we must have $S_* > -1$.

Step 4: Adjusting to initial data and right hand side. For $|\nu|$ and θ such that

$$|\nu| \leq (\beta + 1)^{\frac{\beta}{\beta+1}} T^{-\frac{3\beta+1}{\beta+1}}, \quad |\theta^{-\frac{3\beta+1}{\beta+1}} - 1| \leq B_*^{-1} T^{-1}$$

we set

$$\hat{Y}_p(t; \nu, \theta) = \theta^{-\frac{1}{\beta+1}} \tilde{Y}_p(\theta(t - \nu)).$$

Then, \hat{Y} solves the same equation as \tilde{Y} and with (4.31) we find

$$\begin{aligned} \hat{Y}(-T; \nu, \theta) &- (\beta + 1)^{\frac{1}{\beta+1}} T^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - B_* T^{-\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \left[\nu(\beta + 1)^{-\frac{\beta}{\beta+1}} T^{-\frac{\beta}{\beta+1}} + O\left(\nu^2 T^{-\frac{\beta}{\beta+1}-1}\right) \right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\quad + \left[\left(\theta^{-\frac{3\beta+1}{\beta+1}} - 1 \right) B_* T^{-\frac{3\beta}{\beta+1}} + O\left(\nu \theta^{-\frac{3\beta+1}{\beta+1}} T^{-\frac{3\beta}{\beta+1}-1}\right) \right] \begin{pmatrix} 1 \\ -1 \end{pmatrix} + O\left(T^{-\frac{3\beta}{\beta+1}-1}\right) \\ &= O\left(T^{-\frac{3\beta}{\beta+1}-1}\right). \end{aligned} \quad (4.33)$$

Moreover, the vanishing times of \hat{Y} are $\hat{\tau}_1 = \nu + \theta(S_* + 1)$ and $\hat{\tau}_2 = \nu + \theta(S_* - 1)$.

Let now Y_1, Y_2 be a solution of

$$\dot{Y}_1 = -2Y_1^{-\beta} + F_1$$

in $(\hat{\tau}_2, \hat{\tau}_1)$ with terminal data $Y_1(\hat{\tau}_1) = 0$ and of (4.18) in $(-\infty, \hat{\tau}_2)$ with $Y_2(\hat{\tau}_2) = 0$ and $Y_1(\hat{\tau}_2)$ given by its evolution in $(\hat{\tau}_2, \hat{\tau}_1)$. If η_0 is sufficiently small, existence and uniqueness follow from the same arguments as in Step 1, and a Gronwall argument implies

$$|Y_p(t) - \hat{Y}_p(t)| \leq C(-t)\eta$$

for $t < \hat{\tau}_2$, because Y_p and \hat{Y}_p behave like a power law near their vanishing time and are bounded from below once they have reached a certain size depending on β and η . Using (4.33), we therefore find

$$Y(-T) - (\beta + 1)^{\frac{1}{\beta+1}} T^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - B_* T^{-\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = O\left(\eta T + T^{-\frac{3\beta}{\beta+1}-1}\right).$$

The claim now follows from the inequality between η and T and from the fact that varying ν and θ sweeps a neighborhood of $[(\beta + 1)T]^{1/(\beta+1)}(1, 1) - B_* T^{-3\beta/(\beta+1)}(1, -1)$. \square

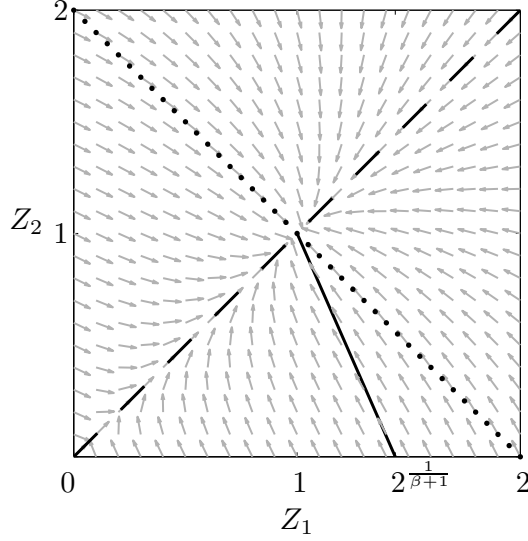


Figure 4.1: Phase portrait for (4.32) and $\beta \geq \beta_*$. Every solution $(Z_1, Z_2)(s)$ of (4.32) approaches the point $(1, 1)$ as $s \rightarrow \infty$ either along the dashed line or the dotted line. In particular, the solution constructed in the proof of Lemma 4.7 runs along the dotted line. This implies $S_* > -1$ because for $S \leq -1$ the initial values are in the region left of the solid line connecting $(1, 1)$ and $(2^{1/(\beta+1)}, 0)$, where the dynamics of (4.32) lead to the wrong asymptotics.

Remark. Showing that $S_* > -1$ is the only part of the non-uniqueness result where we use $\beta \geq \beta_*$. Indeed, our argument breaks down for $\beta < \beta_*$, because near the axis $Z_2 = 0$ the dynamics of (4.32) do not point to the left of the line connecting $(1, 1)$ and $(2^{1/(\beta+1)}, 0)$. However, formal considerations as well as the phase portrait close to $(1, 1)$ indicate that still only trajectories for $S_* > -1$ reach the point $(1, 1)$ along the correct direction $(-1, 1)$, yet a proof seems to require a more detailed investigation of the phase portrait, which is beyond the scope of the paper.

Our final auxiliary result concerns the solution of an ODE that will appear as linearization of the original equation.

Lemma 4.8 (Solution of linear equation). *For $0 \leq T < \bar{\tau}$ let $Y = (Y_1, Y_2): [T, \bar{\tau}) \rightarrow \mathbb{R}^2$ solve the equation*

$$\frac{d}{dt} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \frac{\beta}{\beta+1} \frac{1}{\bar{\tau}-t} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \quad (4.34)$$

with initial data $Y(T) = Y^0 \in \mathbb{R}^2$ and continuous $F = (F_1, F_2): [T, \bar{\tau}) \rightarrow \mathbb{R}^2$. Then the following statements are true.

1. *If $F_1 \equiv F_2 \equiv 0$, then*

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} (t) = \Phi(t; T, \bar{\tau}) Y^0,$$

where

$$\Phi(t; T, \bar{\tau}) = \frac{1}{2} \left(\frac{\bar{\tau}-T}{\bar{\tau}-t} \right)^{\frac{\beta}{\beta+1}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \left(\frac{\bar{\tau}-T}{\bar{\tau}-t} \right)^{\frac{3\beta}{\beta+1}} \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}.$$

2. *If $Y^0 = (0, 0)$ and F_1, F_2 are constant functions, then*

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} (t) = (\bar{\tau}-t)A - (\bar{\tau}-T)\Phi(t; T, \bar{\tau})A$$

where $A = (A_1, A_2)$ is the solution of

$$-\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{\beta}{\beta+1} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

3. If $Y^0 = (0, 0)$, then

$$|Y_p(t)| \leq C \int_T^t \left(\frac{\bar{\tau} - s}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} (|F_1(s)| + |F_2(s)|) ds$$

for $p = 1, 2$ and all $t \in [T, \bar{\tau})$.

Proof. It is easily checked that $\Phi(t; T, \bar{\tau})$ is a fundamental matrix for the homogeneous linear equation associated with (4.34) and that $\Phi(T; T, \bar{\tau})$ is the identity matrix. Hence, the first claim follows immediately and the second by a direct computation. Finally, the variation of constants formula

$$Y(t) = \int_T^t \Phi(t; T, \bar{\tau}) \Phi(s; T, \bar{\tau})^{-1} F(s) ds = \int_T^t \Phi(t; s, \bar{\tau}) F(s) ds$$

implies the third assertion. \square

4.4 Iterative estimates

The aim of this section is to derive the main iterative estimates for the non-uniqueness result. To this end, let x^N and \bar{x}^N be solutions to the truncated problem with initial data with and without some $\varepsilon^N \in (-1/8, 1/8)$, respectively. For simplicity of notation we drop the upper index N .

We denote by τ_k and $\bar{\tau}_k$, $k = -3N, \dots, 3N$ the vanishing times of x and \bar{x} , respectively, and define

$$T_j = \max \{ \tau_{3j+1}, \tau_{3j+2}, \bar{\tau}_{3j+1} \}$$

for $j = 0, \dots, N-1$; for convenience we set $T_N = 0$. Lemma 4.3 immediately yields

$$cR_j^{\beta+1} \leq T_j \leq CR_j^{\beta+1},$$

for $j > N_*$, and by Proposition 4.5, the remark thereafter and the definition of (R_m) we have

$$\min \{ \tau_{3(j-1)+1}, \tau_{3(j-1)+2}, \bar{\tau}_{3(j-1)+1} \} - T_j \geq cR_j^{\beta+1} \quad (4.35)$$

as well as

$$\frac{\min \{ \tau_{3(j-1)+1}, \tau_{3(j-1)+2}, \bar{\tau}_{3(j-1)+1} \}}{T_j} \geq c > 1$$

for $j > N_*$. As a consequence, the vanishing times of $3(j-1) + p$ and $3j + p$ are separated from each other and we may consider non-adjacent small particles separately.

In the following we restrict our attention to the time interval $[T_j, T_{j-1}]$ and we refer to the vanishing particles $3(j-1) + p$ and their neighbors $3(j-1)$, $3j$ as the *active particles*; the others we call *inactive particles*. To study the change of $x - \bar{x}$ we group the inactive particles according to size and location and set

$$\begin{aligned} \|x - \bar{x}\|_{s,j} &:= \sup_{\substack{N_* \leq k \leq j-2 \\ p=1,2}} \frac{|x_{3k+p} - \bar{x}_{3k+p}|}{R_k^{\beta+1}}, \\ \|x - \bar{x}\|_{l,j} &:= \sup_{N_* \leq k \leq j-2} |x_{3k} - \bar{x}_{3k}|, \\ \|x - \bar{x}\|_{r,j} &:= \sup_{j+1 \leq k \leq N} |x_{3k} - \bar{x}_{3k}|. \end{aligned}$$

Moreover, we denote by

$$D_{p,j} := x_{3(j-1)+p} - \bar{x}_{3(j-1)+p}, \quad p = 0, 1, 2, 3$$

the differences of the active particles.

The following theorem is the key result of this section and summarizes the propagation of a perturbation from the particles $3j$ at time T_j to the particles $3(j-1)$ in the time interval $[T_j, T_{j-1}]$.

Theorem 4.9 (Iterative estimates). *Let $\beta \geq \beta_*$ where β_* is as in (4.2). There exist $\delta_* \in (0, 1)$, $N_* \in \mathbb{N}$, $\gamma_* \in (0, 1/3)$, $a_* > 0$ and $C_* > 0$ such that for all $N > N_*$, $\delta \in (0, \delta^*)$ and $\gamma \in (0, \gamma^*)$ the following property is true. If for some $j = N_* + 1, \dots, N-1$ the particles $3j$ satisfy*

$$|D_{3,j}(T_j)| = \delta \tag{4.36}$$

and the other particles are estimated by

$$|D_{0,j}(T_j)| \leq \delta\delta_*, \quad |D_{1,j}(T_j)| \leq R_{j-1}^{\beta+1}\delta\delta_*, \quad |D_{2,j}(T_j)| \leq R_{j-1}^{\beta+1}\delta\delta_* \tag{4.37}$$

as well as

$$\|x - \bar{x}\|_{s,j}(T_j) \leq \delta\delta_*, \quad \|x - \bar{x}\|_{l,j}(T_j) \leq \delta\delta_*, \quad \|x - \bar{x}\|_{r,j}(T_j) \leq 2\delta, \tag{4.38}$$

then there is

$$\bar{\delta} = \bar{\delta}(\delta) \in \left[\frac{3}{4}a_* \left(R_{j-1}^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}}, \frac{5}{4}a_* \left(R_{j-1}^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} \right]$$

such that we have

$$|D_{3,j-1}(T_{j-1})| = |D_{0,j}(T_{j-1})| = \bar{\delta}$$

and

$$|D_{0,j-1}(T_{j-1})| \leq \bar{\delta}\delta_*, \quad |D_{1,j-1}(T_{j-1})| \leq R_{j-2}^{\beta+1}\bar{\delta}\delta_*, \quad |D_{2,j-1}(T_{j-1})| \leq R_{j-2}^{\beta+1}\bar{\delta}\delta_*$$

as well as

$$\|x - \bar{x}\|_{s,j-1}(T_{j-1}) \leq \bar{\delta}\delta_*, \quad \|x - \bar{x}\|_{l,j-1}(T_{j-1}) \leq \bar{\delta}\delta_*, \quad \|x - \bar{x}\|_{r,j-1}(T_{j-1}) \leq 2\bar{\delta},$$

provided that $C_*\delta \leq \bar{\delta}$.

For the rest of the section we drop the index j wherever it is not necessary and write $\|\cdot\|_s$, D_p and so on. Moreover, it is convenient to abbreviate

$$D_0^\beta := x_{3(j-1)}^{-\beta} - \bar{x}_{3(j-1)}^{-\beta} \quad \text{and} \quad D_3^\beta := x_{3j}^{-\beta} - \bar{x}_{3j}^{-\beta}.$$

If not stated otherwise, we also assume $N_* \leq j < N$ where N_* is sufficiently large so that all previous results apply.

The next lemma bounds the change of particles up to time T_{j-1} , and Lemmas 4.11–4.12 provide a few simple estimates for the particle differences.

Lemma 4.10 (Constancy of left inactive particles and all large particles). *For all $t \in [0, T_{j-1}]$ we have*

$$\begin{aligned} |x_{3k+p}(t) - x_{3k+p}(0)| &\leq C\gamma^{j-1-k}R_k & \text{for } k = N_*, \dots, j-2, \quad p = 0, 1, 2, \\ |x_{3k}(t) - x_{3k}(0)| &\leq CR_{j-1} & \text{for } k = j-1, \dots, N-1. \end{aligned}$$

In particular, choosing γ sufficiently small, we obtain

$$x_{3k+p}(t) \geq \frac{1}{4}R_k$$

for all $t \in [0, T_{j-1}]$ and $k = N_*, \dots, j-2$. Similar inequalities hold for \bar{x} .

Proof. From Lemma 4.3 and inequality (4.35) we know that for all $t \in [0, T_{j-1}]$ we have

$$x_{3k+p}(t) \geq c(\tau_{3k+p} - T_{j-1})^{\frac{1}{\beta+1}} \geq cR_{j-1}$$

for $k = N_*, \dots, j-2$ and $p = 1, 2$. Since clearly also

$$x_{3k}(t) \geq \frac{1}{2} \geq R_{j-1}$$

for $k = 0, \dots, j-1$, we conclude that

$$|x_{3k+p}(t) - x_{3k+p}(0)| \leq \int_0^{T_{j-1}} |\dot{x}_{3k+p}(s)| ds \leq CR_{j-1}^{-\beta} T_{j-1} \leq CR_{j-1} \leq C\gamma^{j-1-k} R_k$$

for $k = N_*, \dots, j-2$ and $p = 0, 1, 2$. This proves the first inequality.

For the third note that up to time T_{j-1} each particle x_{3k} , $k \geq j$ has seen at most four different small neighbors whose contributions to its velocity are of size

$$\int_0^\tau y^{-\beta} dt \leq C \int_0^\tau (\tau - t)^{-\frac{\beta}{\beta+1}} dt \leq C\tau^{\frac{1}{\beta+1}}$$

where y denotes a neighbor with vanishing time τ . Since moreover large particles remain large throughout the evolution we deduce

$$|x_{3k}(t) - x_{3k}(0)| \leq C \left(R_{j-1}^{\beta+1} + R_k + R_{k-1} \right) \leq CR_{j-1}$$

for all $t \in [0, T_{j-1}]$ and $k \geq j$. Finally, for $k = 3(j-1)$ the particles $x_{3(j-1)+p}$, $p = 1, 2$ vanish before the time T_{j-1} , whereas we have $x_{3(j-2)+2} \geq cR_{j-1}$. Thus, combining the two arguments from above finishes the proof for x , and the proof for \bar{x} is identical. \square

Lemma 4.11 (Estimates for inactive particles). *We have*

$$\|x - \bar{x}\|_s(t) \leq C\|x - \bar{x}\|_s(T_j) + C \frac{t - T_j}{R_{j-2}^{\beta+1}} \|x - \bar{x}\|_l(T_j) + \frac{C}{R_{j-2}^{\beta+1}} \int_{T_j}^t |D_0^\beta(s)| ds, \quad (4.39)$$

$$\|x - \bar{x}\|_l(t) \leq C\|x - \bar{x}\|_l(T_j) + C(t - T_j)\|x - \bar{x}\|_s(T_j) + C\gamma^{\beta+1} \int_{T_j}^t |D_0^\beta(s)| ds \quad (4.40)$$

and

$$\|x - \bar{x}\|_r(t) \leq C\|x - \bar{x}\|_r(T_j) + C \int_{T_j}^t |D_3^\beta(s)| ds \quad (4.41)$$

for all $t \in [T_j, T_{j-1}]$.

Proof. From the Constancy Lemma 4.10 we know that up to the time T_{j-1} the large and small particles on the left of the active ones remain close to their initial value 1 and R_k , respectively. In the difference of the equations for x and \bar{x} we may thus linearize and find

$$|x_{3k}^{-\beta} - \bar{x}_{3k}^\beta| \leq C|x_{3k} - \bar{x}_{3k}| \leq C\|x - \bar{x}\|_l$$

as well as

$$|x_{3k+p}^{-\beta} - \bar{x}_{3k+p}^\beta| \leq C \frac{|x_{3k+p} - \bar{x}_{3k+p}|}{R_k^{\beta+1}} \leq C\|x - \bar{x}\|_s$$

for $k = N_*, \dots, 3(j-2)$ and $p = 1, 2$. Consequently, we obtain

$$\frac{d}{dt} \|x - \bar{x}\|_l \leq C\|x - \bar{x}\|_l + C\|x - \bar{x}\|_s \quad (4.42)$$

and

$$\frac{d}{dt}\|x - \bar{x}\|_s \leq \frac{C}{R_{j-2}^{\beta+1}}\|x - \bar{x}\|_l + \frac{C}{R_{j-2}^{\beta+1}}\|x - \bar{x}\|_s + \frac{1}{R_{j-2}^{\beta+1}}|D_0^\beta(s)| \quad (4.43)$$

for all $t \in [T_j, T_{j-1}]$, where the last term in (4.43) stems from the equation for $3(j-2) + 2$. Applying Gronwall's inequality to (4.42) and using $t - T_j \leq 1$ to incorporate the resulting exponential terms into the constants, we infer that

$$\|x - \bar{x}\|_l(t) \leq C\|x - \bar{x}\|_l(T_j) + C \int_{T_j}^t \|x - \bar{x}\|_s(s) ds. \quad (4.44)$$

Similarly, we deduce from (4.43) and $t - T_j \leq R_{j-1}^{\beta+1} \leq R_{j-2}^{\beta+1}$ that

$$\|x - \bar{x}\|_s(t) \leq C\|x - \bar{x}\|_s(T_j) + \frac{C}{R_{j-2}^{\beta+1}} \int_{T_j}^t \|x - \bar{x}\|_l(s) + |D_0^\beta(s)| ds, \quad (4.45)$$

and combining (4.44), (4.45) and

$$\int_{T_j}^t \int_{T_j}^s \|x - \bar{x}\|_s(r) dr ds = \int_{T_j}^t (t - r) \|x - \bar{x}\|_s(r) dr \leq R_{j-1}^{\beta+1} \int_{T_j}^t \|x - \bar{x}\|_s(r) dr$$

we arrive at

$$\begin{aligned} \|x - \bar{x}\|_s(t) &\leq C\|x - \bar{x}\|_s(T_j) + C \frac{t - T_j}{R_{j-2}^{\beta+1}} \|x - \bar{x}\|_l(T_j) \\ &\quad + C \gamma^{\beta+1} \int_{T_j}^t \|x - \bar{x}\|_s(s) ds + \frac{C}{R_{j-2}^{\beta+1}} \int_{T_j}^t |D_0^\beta(s)| ds. \end{aligned}$$

Now, a Gronwall argument for $t \mapsto \int_{T_j}^t \|x - \bar{x}\| ds$ proves (4.39). The inequalities (4.40) and (4.41) are derived analogously. \square

Lemma 4.12 (Estimates for active large particles I). *We have*

$$\begin{aligned} |D_0(t)| &\leq C|D_0(T_j)| + C(t - T_j) \left(\|x - \bar{x}\|_s(T_j) + \gamma^{\beta+1} \|x - \bar{x}\|_l(T_j) \right) \\ &\quad + C \int_{T_j}^t |x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta}|(s) ds \quad (4.46) \end{aligned}$$

and

$$|D_3(t)| \leq C|D_3(T_j)| + C(t - T_j) \|x - \bar{x}\|_r(T_j) + C \int_{T_j}^t |x_{\sigma_-(3j)}^{-\beta} - \bar{x}_{\sigma_-(3j)}^{-\beta}|(s) ds \quad (4.47)$$

for all $t \in [T_j, T_{j-1}]$.

Proof. Similar to the proof of the previous lemma we linearize in the derivative of D_0 to find

$$\frac{d}{dt}|D_0| \leq C\|x - \bar{x}\|_s + C|D_0| + |x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta}|.$$

Note that $\sigma_+(3(j-1))$ may take the values $3(j-1) + p$, $p = 1, 2, 3$ in the time interval $[T_j, T_{j-1}]$ and may be different for x and \bar{x} . With Gronwall's inequality we get

$$|D_0(t)| \leq C|D_0(T_j)| + C \int_{T_j}^t \|x - \bar{x}\|_s ds + C \int_{T_j}^t |x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta}| ds, \quad (4.48)$$

and from (4.39) with $|D_0^\beta| \leq C|D_0|$ we obtain

$$\begin{aligned} \int_{T_j}^t \|x - \bar{x}\|_s(s) \, ds &\leq C(t - T_j) \left(\|x - \bar{x}\|_s(T_j) + C\gamma^{\beta+1} \|x - \bar{x}\|_l(T_j) \right) \\ &\quad + C\gamma^{\beta+1} \int_{T_j}^t |D_0(s)| \, ds. \end{aligned} \quad (4.49)$$

Combining (4.49) with (4.48) and applying Gronwall's inequality to $t \mapsto \int_{T_j}^t |D_0(s)| \, ds$ proves (4.46). The derivation of (4.47) is similar. \square

Next, we consider the active small particles, and our aim is to derive an approximate solution formula for D_p , $p = 1, 2$. However, since the equation for D_p contains D_0^β or D_3^β , we need precise estimates for the latter.

Lemma 4.13 (Estimates for D_0^β and D_3^β). *Suppose that*

$$|D_p(t)| \leq \frac{1}{2} \bar{x}_{3(j-1)+p}(t), \quad p = 1, 2 \quad (4.50)$$

in some time interval $[T_j, T_]$ where $T_j < T_* \leq \min \{ \tau_{3(j-1)+1}, \tau_{3(j-1)+2}, \bar{\tau}_{3(j-1)+1} \}$. Then we have*

$$\begin{aligned} |D_0^\beta(t) - D_0^\beta(T_j)| &\leq C(t - T_j) \left(\|x - \bar{x}\|_s(T_j) + C\gamma^{\beta+1} \|x - \bar{x}\|_l(T_j) \right) \\ &\quad + CR_{j-1} |D_0^\beta(T_j)| + C \int_{T_j}^t \frac{|D_1(s)| \, ds}{\bar{\tau}_{3(j-1)+1} - s} \end{aligned} \quad (4.51)$$

and

$$|D_3^\beta(t) - D_3^\beta(T_j)| \leq C(t - T_j) \|x - \bar{x}\|_r(T_j) + CR_{j-1} |D_3^\beta(T_j)| + C \int_{T_j}^t \frac{|D_2(s)| \, ds}{\bar{\tau}_{3(j-1)+2} - s} \quad (4.52)$$

for all $t \in [T_j, T_*]$.

Proof. In

$$\frac{d}{dt} D_0^\beta = -\beta \bar{x}_{3(j-1)}^{-(\beta+1)} \frac{d}{dt} D_0 + \beta \left(\bar{x}_{3(j-1)}^{-(\beta+1)} - x_{3(j-1)}^{-(\beta+1)} \right) \dot{x}_{3(j-1)} \quad (4.53)$$

we estimate the first term on the right hand side by

$$\begin{aligned} \left| \beta \bar{x}_{3(j-1)}^{-(\beta+1)} \frac{d}{dt} D_0 \right| &\leq C \|x - \bar{x}\|_s + C |D_0^\beta| + C \left| x_{3(j-1)+1}^{-\beta} - \bar{x}_{3(j-1)+1}^{-\beta} \right| \\ &\leq C \|x - \bar{x}\|_s + C |D_0^\beta| + C \frac{|D_1|}{\bar{x}_{3(j-1)+1}^{\beta+1}}, \end{aligned}$$

using that (4.50) implies $x_{3(j-1)+1} \geq \bar{x}_{3(j-1)+1}/2$ and thus

$$\left| x_{3(j-1)+1}^{-\beta} - \bar{x}_{3(j-1)+1}^{-\beta} \right| \leq C \frac{|D_1|}{\bar{x}_{3(j-1)+p}^{\beta+1}}.$$

In the second term of (4.53) we use

$$\left| \bar{x}_{3(j-1)}^{-(\beta+1)} - x_{3(j-1)}^{-(\beta+1)} \right| = \left| \left(\bar{x}_{3(j-1)}^{-\beta} \right)^{\frac{\beta+1}{\beta}} - \left(x_{3(j-1)}^{-\beta} \right)^{\frac{\beta+1}{\beta}} \right| \leq C |D_0^\beta|,$$

where we have linearized $s \mapsto s^{(\beta+1)/\beta}$, and

$$|\dot{x}_{3(j-1)}| \leq C \bar{x}_{3(j-1)+1}^{-\beta}$$

noting that up to time T_{j-1} the particles $x_{3(j-1)}$ and $x_{3(j-2)+2}$ remain large compared to $x_{3(j-1)+1}$ due to the Constancy Lemma 4.10. As a consequence, (4.53) yields

$$\left| \frac{d}{dt} D_0^\beta \right| \leq C \left(\|x - \bar{x}\|_s + \frac{|D_1|}{\bar{x}_{3(j-1)+1}^{\beta+1}} + \frac{|D_0^\beta|}{\bar{x}_{3(j-1)+1}^\beta} \right)$$

and after rewriting this estimate as

$$\left| \frac{d}{dt} (D_0^\beta - D_0^\beta(T_j)) \right| \leq C \left(\|x - \bar{x}\|_s + \frac{|D_1|}{\bar{x}_{3(j-1)+1}^{\beta+1}} + \frac{|D_0^\beta - D_0^\beta(T_j)|}{\bar{x}_{3(j-1)+1}^\beta} + \frac{|D_0^\beta(T_j)|}{\bar{x}_{3(j-1)+1}^\beta} \right)$$

we use Gronwall's inequality to obtain

$$\begin{aligned} & |D_0^\beta(t) - D_0^\beta(T_j)| \\ & \leq C \int_{T_j}^t \left(\|x - \bar{x}\|_s + \frac{|D_1(s)|}{\bar{x}_{3(j-1)+1}(s)^{\beta+1}} + \frac{|D_0^\beta(T_j)|}{\bar{x}_{3(j-1)+1}(s)^\beta} \right) \exp \left(\int_s^t \frac{C \, dr}{\bar{x}_{3(j-1)+1}(r)^\beta} \right) ds. \end{aligned}$$

Since the lower bound (4.13) implies

$$\int_{T_j}^t \frac{ds}{\bar{x}_{3(j-1)+1}(s)^\beta} \leq C \int_{T_j}^t (\bar{\tau}_{3(j-1)+1} - t)^{-\frac{\beta}{\beta+1}} ds \leq (\bar{\tau}_{3(j-1)+1} - T_j)^{\frac{1}{\beta+1}} \leq C R_{j-1},$$

we conclude

$$|D_0^\beta(t) - D_0^\beta(T_j)| \leq C \int_{T_j}^t \left(\|x - \bar{x}\|_s(s) + \frac{|D_1(s)|}{\bar{x}_{3(j-1)+1}(s)^{\beta+1}} \right) ds + C R_{j-1} |D_0^\beta(T_j)|.$$

Using now (4.39) for $\|x - \bar{x}\|_s$ and Gronwall's inequality for $t \mapsto \int_{T_j}^t |D_0^\beta(s) - D_0^\beta(T_j)| ds$ finishes the proof of (4.51). The proof of (4.52) is similar. \square

Lemma 4.14 (Linearized equation for active small particles). *Suppose that (4.50) holds up to time $T_* > T_j$. Then we have*

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} - \frac{\beta}{\beta+1} \frac{1}{\bar{\tau}_{3(j-1)+1} - t} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \\ = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} D_0^\beta - D_0^\beta(T_j) \\ D_3^\beta - D_3^\beta(T_j) \end{pmatrix} + \begin{pmatrix} D_0^\beta(T_j) \\ D_3^\beta(T_j) \end{pmatrix} \end{aligned} \quad (4.54)$$

where

$$|\omega_p| \leq C \frac{|D_1|^2 + |D_2|^2}{(\bar{\tau}_{3(j-1)+1} - t)^{\frac{\beta+2}{\beta+1}}} + C \frac{|D_1| + |D_2|}{(\bar{\tau}_{3(j-1)+1} - t)^{\frac{1}{\beta+1}}}, \quad p = 1, 2$$

for all $t \in [T_j, T_*]$.

Proof. We employ (4.50) to linearize the differences $x_{3(j-1)+p}^{-\beta} - \bar{x}_{3(j-1)+p}^{-\beta}$ around $\bar{x}_{3(j-1)+p}$ in

$$\frac{d}{dt} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_{3(j-1)+1}^{-\beta} - \bar{x}_{3(j-1)+1}^{-\beta} \\ x_{3(j-1)+2}^{-\beta} - \bar{x}_{3(j-1)+2}^{-\beta} \end{pmatrix} + \begin{pmatrix} D_0^\beta \\ D_3^\beta \end{pmatrix},$$

which is

$$x_{3(j-1)+p}^{-\beta} - \bar{x}_{3(j-1)+p}^{-\beta} = -\beta \frac{D_p}{\bar{x}_{3(j-1)+p}^{\beta+1}} + O\left(\frac{D_p^2}{\bar{x}_{3(j-1)+p}^{\beta+2}}\right).$$

Then, replacing $\bar{x}_{3(j-1)+p}^{-(\beta+1)}$ with the power law obtained in Lemma 4.6, that is

$$\bar{x}_{3(j-1)+p}^{-(\beta+1)} = \frac{1}{\beta+1} \frac{1}{\bar{\tau}_{3(j-1)+p} - t} + O\left((\bar{\tau}_{3(j-1)+p} - t)^{-\frac{1}{\beta+1}}\right),$$

and using the lower bound (4.13) to get

$$\bar{x}_{3(j-1)+p}^{-(\beta+2)} \leq C (\bar{\tau}_{3(j-1)+p} - t)^{-\frac{\beta+2}{\beta+1}}$$

proves the claim. \square

Proposition 4.15 (Approximate solution for active small particles). *Given $\delta_* \in (0, 1)$ and $\delta \in (0, \delta_*)$ there are constants $N_* = N_*(\beta, \delta_*)$ and $C = C(\beta, \delta_*)$ with the following property. If (4.36)–(4.38) are satisfied for some $j > N_*$ then we have*

$$\tau_{3(j-1)+p} > \bar{\tau}_{3(j-1)+1} - C \left(R_{j-1}^{4\beta+1} \delta \right)^{\frac{\beta+1}{3\beta+1}} =: \tau_{j-1}^*.$$

Moreover, (4.50) holds for all $t \in [T_j, \tau_{j-1}^*]$ and we have

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} (t) = -\frac{D_3(T_j)}{\bar{x}_{3j}(T_j)^{\beta+1}} \phi(t; T_j, \bar{\tau}_{3(j-1)+1}) + \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} (t)$$

where

$$|\rho_p(t)| \leq C R_{j-1}^{\beta+1} \delta \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{3\beta}{\beta+1}} \delta_*$$

and

$$\begin{aligned} \phi(t; T_j, \bar{\tau}_{3(j-1)+1}) &= -(\bar{\tau}_{3(j-1)+1} - t) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \\ &\quad + C_3 (\bar{\tau}_{3(j-1)+1} - T_j) \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{\beta}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &\quad - C_4 (\bar{\tau}_{3(j-1)+1} - T_j) \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

with positive constants C_1, C_2, C_3, C_4 .

Proof. The Constancy Lemma 4.10 for time T_j and $k = j-1$ yields $\bar{x}_{3(j-1)+p}(T_j) \geq R_{j-1}/2$, whereas (4.37) implies $|D_p(T_j)| \leq R_{j-1}/8$ for $p = 1, 2$, all $\delta < \delta_* < 1$ and $j > N_*$ provided that we choose N_* sufficiently large. Thus, (4.50) is true for some $T_* \in (T_j, \bar{\tau}_{3(j-1)+1})$ and Lemmas 4.13 and 4.14 apply. Moreover, (4.37) also provides

$$|D_1(t)| + |D_2(t)| \leq M R_{j-1}^{\beta+1} \delta \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{3\beta}{\beta+1}} \quad (4.55)$$

for $t \in [T_j, T_*]$ and some constant $M > 2$ that will be chosen below.

We now proceed to solve the linearized equation (4.54) separately for the different contributions on its right hand side and the initial data. Then, we determine the time up to which our approximation is valid and the leading order terms of the solution. To simplify the notation, we abbreviate

$$\bar{\tau} = \bar{\tau}_{3(j-1)+1}, \quad T = T_j \quad \text{and} \quad R = R_{j-1}$$

throughout the proof.

Step 1: Initial data $D_p(T)$. According to Lemma 4.8(1) the contribution of the initial data $D_p(T_j)$ to the solution of (4.54) with right hand side equal to zero is given by

$$Z^{ID}(t) = \Phi(t; T, \bar{\tau}) \begin{pmatrix} D_1(T) \\ D_2(T) \end{pmatrix}.$$

By assumption (4.37) and the formula for Φ this is easily estimated as

$$|Z_1^{ID}| + |Z_2^{ID}| \leq 2 \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} (|D_1(T)| + |D_2(T)|) \leq 4R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \delta_*.$$

Step 2: Right hand side $(D_0^\beta(T), D_3^\beta(T))$. Here we use Lemma 4.8(2) separately for $(D_0^\beta(T), 0)$ and $(0, D_3^\beta(T))$. Calling the solutions Z^{R1} and Z^{R2} , respectively, we obtain after a straightforward computation that

$$|Z_p^{R1}| \leq CR^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \delta_* \quad \text{and} \quad |Z_p^{R2}| \leq CR^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}}.$$

Step 3: Right hand side $(D_0^\beta(t) - D_0^\beta(T), D_3^\beta(t) - D_3^\beta(T))$. Using the assumptions (4.38) and (4.36)–(4.37) for $p = 0, 3$ as well as the estimate (4.55), we obtain from Lemma 4.13 that

$$\begin{aligned} |D_0^\beta(t) - D_0^\beta(T)| &\leq CR\delta\delta_* + CMR^{\beta+1}\delta \int_{T_j}^t \left(\frac{\bar{\tau} - T}{\bar{\tau} - s} \right)^{\frac{3\beta}{\beta+1}} \frac{ds}{\bar{\tau} - s} \\ &\leq CR\delta\delta_* + CMR^{\beta+1}\delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \end{aligned}$$

and

$$|D_3^\beta(t) - D_3^\beta(T)| \leq CR\delta + CMR^{\beta+1}\delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}}.$$

By Lemma 4.8(3) the corresponding solution Z^{R0} may thus be estimated as

$$\begin{aligned} |Z_p^{R0}| &\leq C\delta \int_T^t \left(\frac{\bar{\tau} - s}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \left(R + MR^{\beta+1} \left(\frac{\bar{\tau} - T}{\bar{\tau} - s} \right)^{\frac{3\beta}{\beta+1}} \right) ds \\ &\leq CR^{\beta+1}\delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} (R + MR^{\beta+1}). \end{aligned}$$

Step 4: Right hand side ω_p . Plugging (4.55) into the estimate for ω_p from Lemma 4.14 yields

$$|\omega_p(s)| \leq \frac{CM^2R^{2(\beta+1)}\delta^2}{(\bar{\tau} - s)^{\frac{\beta+2}{\beta+1}}} \left(\frac{\bar{\tau} - T}{\bar{\tau} - s} \right)^{\frac{6\beta}{\beta+1}} + \frac{CMR^{\beta+1}\delta}{(\bar{\tau} - s)^{\frac{1}{\beta+1}}} \left(\frac{\bar{\tau} - T}{\bar{\tau} - s} \right)^{\frac{3\beta}{\beta+1}}.$$

Considering both terms separately, we deduce from Lemma 4.8(3) that the associated solutions $Z^{\omega 1}$ and $Z^{\omega 2}$ satisfy

$$\begin{aligned} |Z_p^{\omega 1}| &\leq CM^2R^{2(\beta+1)}\delta^2 \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \int_T^t (\bar{\tau} - s)^{-\frac{4\beta+2}{\beta+1}} ds \\ &\leq CM^2R^{2(\beta+1)}\delta^2 \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{6\beta}{\beta+1}} \frac{1}{(\bar{\tau} - t)^{\frac{1}{\beta+1}}} \end{aligned}$$

and

$$\begin{aligned} |Z_p^{\omega 2}| &\leq CM R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \int_T^t (\bar{\tau} - s)^{-\frac{1}{\beta+1}} ds \\ &\leq CM R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} R^\beta. \end{aligned}$$

Step 5: Continuation. From the previous estimates we find that (D_1, D_2) satisfies

$$\begin{aligned} |D_p| &\leq |Z_p^{ID}| + |Z_p^{R1}| + |Z_p^{R2}| + |Z_p^{R0}| + |Z_p^{\omega 1}| + |Z_p^{\omega 2}| \\ &\leq R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} C \left(\delta_* + 1 + R + MR^{\beta+1} + MR^\beta \right) + |Z_p^{\omega 1}|. \end{aligned}$$

Since $\delta_* < 1$ we may choose $N_* \in \mathbb{N}$ and $M > 0$, both depending only on β , such that

$$C \left(\delta_* + 1 + R + MR^{\beta+1} + MR^\beta \right) \leq \frac{M}{4}$$

for all $j > N_*$. Moreover, we can estimate $|Z_p^{\omega 1}|$ similarly by νM , where $0 < \nu < 1/4$ will be chosen below, provided that

$$CM R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \frac{1}{(\bar{\tau} - t)^{\frac{1}{\beta+1}}} \leq \nu, \quad (4.56)$$

and due to $\bar{\tau} - T \leq CR^{\beta+1}$ this inequality is true if

$$\bar{\tau} - t \geq \left(\frac{CM}{\nu} R^{4\beta+1} \delta \right)^{\frac{\beta+1}{3\beta+1}}. \quad (4.57)$$

Hence, for such $t \geq T$ also inequality (4.55) holds and together with (4.56) implies

$$|D_1| + |D_2| \leq \nu (\bar{\tau} - t)^{\frac{1}{\beta+1}}.$$

As a consequence, also (4.50) and Steps 1–4 are valid up to the time τ_{j-1}^* defined by equality in (4.57) if ν is sufficiently small.

Step 6: Solution formula for (D_1, D_2) . Now all contributions to (D_1, D_2) except for Z_p^{R2} are estimated by

$$CM R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \left(\delta_* + R + MR^{\beta+1} + \nu + MR^\beta \right).$$

Therefore, by increasing N_* and decreasing ν – both thus depending on δ_* – we obtain

$$D_p = Z_p^{R2} + \rho_p \quad \text{where} \quad |\rho_p| \leq CM R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}} \delta_*.$$

The solution formula then follows from the linearization

$$D_3^\beta(T) = -\beta \frac{D_3(T)}{\bar{x}_{3j}(T)^{\beta+1}} + O(\delta^2)$$

and Lemma 4.8(2), taking into account that the contribution of $O(\delta^2) \leq O(\delta\delta_*)$ can be estimated similar to Z^{R1} and incorporated into ρ_p . \square

Corollary 4.16. *In the setting of Proposition 4.15, we have*

$$|D_p(t)| \leq CR_{j-1}^{\beta+1} \delta \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{3\beta}{\beta+1}}$$

for $p = 1, 2$ and

$$\begin{aligned} \begin{pmatrix} x_{3(j-1)+1} \\ x_{3(j-1)+2} \end{pmatrix} (t) &= (\beta + 1)^{\frac{1}{\beta+1}} (\bar{\tau}_{3(j-1)+1} - t)^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &+ C_4 \frac{D_3(T_j)}{\bar{x}_{3j}(T_j)^{\beta+1}} (\bar{\tau}_{3(j-1)+1} - T_j) \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &+ O \left((\bar{\tau}_{3(j-1)+1} - t) + R_{j-1}^{\beta+1} \delta \left(\frac{\bar{\tau}_{3(j-1)+1} - T_j}{\bar{\tau}_{3(j-1)+1} - t} \right)^{\frac{\beta}{\beta+1}} + |\rho_1| + |\rho_2| \right) \end{aligned} \quad (4.58)$$

for all $t \in [T_j, \tau_{j-1}^*]$.

Proof. Since $D_3(T_j) = \delta$ the first claim is an immediate consequence of Proposition 4.15, while the second in addition uses

$$\bar{x}_{3(j-1)+p}(t) = (\beta + 1)^{\frac{1}{\beta+1}} (\bar{\tau}_{3(j-1)+1} - t)^{\frac{1}{\beta+1}} + O(\bar{\tau}_{3(j-1)+1} - t),$$

as proved in Lemma 4.6. \square

Corollary 4.17. *Assume that $\beta \geq \beta_*$. In the setting of Proposition 4.15 we have either*

$$\begin{aligned} \tau_{3(j-1)+1} - \bar{\tau}_{3(j-1)+1} &= (S_* + 1) A_* \left(R_{j-1}^{4\beta+1} \delta \right)^{\frac{\beta+1}{3\beta+1}} (1 + o(1)_{R_{j-1} \rightarrow 0}), \\ \tau_{3(j-1)+2} - \bar{\tau}_{3(j-1)+1} &= (S_* - 1) A_* \left(R_{j-1}^{4\beta+1} \delta \right)^{\frac{\beta+1}{3\beta+1}} (1 + o(1)_{R_{j-1} \rightarrow 0}) \end{aligned}$$

if $D_3(T_j) = +\delta$ or the same equations with $\tau_{3(j-1)+1}$ and $\tau_{3(j-1)+2}$ exchanged if $D_3(T_j) = -\delta$. Here

$$A_*^{\frac{3\beta+1}{\beta+1}} = \frac{C_4}{B_*(\beta + 1)^{\frac{4\beta+1}{\beta+1}}} \bar{x}_{3j}(T_j)^{-(\beta+1)} > 0$$

where S_* and B_* are the constants from Lemma 4.7. In particular, since $|S_*| < 1$ we have

$$\begin{aligned} \tau_{3(j-1)+2} &< \bar{\tau}_{3(j-1)+1} < \tau_{3(j-1)+1} & \text{if } D_3(T_j) = +\delta, \\ \tau_{3(j-1)+1} &< \bar{\tau}_{3(j-1)+1} < \tau_{3(j-1)+2} & \text{if } D_3(T_j) = -\delta, \end{aligned}$$

provided that N_* is sufficiently large.

Proof. In the following we abbreviate

$$\bar{\tau} = \bar{\tau}_{3(j-1)+p}, \quad T = T_j \quad \text{and} \quad R = R_{j-1},$$

so that (4.58) reads

$$\begin{aligned} \begin{pmatrix} x_{3(j-1)+1} \\ x_{3(j-1)+2} \end{pmatrix} (t) &= (\beta + 1)^{\frac{1}{\beta+1}} (\bar{\tau} - t)^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_4 \frac{D_3(T)}{\bar{x}_{3j}(T)^{\beta+1}} \frac{(\bar{\tau} - T)^{\frac{4\beta+1}{\beta+1}}}{(\bar{\tau} - t)^{\frac{3\beta}{\beta+1}}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &+ O \left((\bar{\tau} - t) + R^{\beta+1} \delta \left(\frac{\bar{\tau} - T}{\bar{\tau} - t} \right)^{\frac{\beta}{\beta+1}} + |\rho_1| + |\rho_2| \right). \end{aligned}$$

Assuming that $D_3(T) = +\delta$ we let

$$y_p(s) = \frac{x_{3(j-1)+p} \left(\bar{\tau} + A_*(R^{4\beta+1}\delta)^{\frac{\beta+1}{3\beta+1}} s \right)}{A_*^{\frac{1}{\beta+1}} (R^{4\beta+1}\delta)^{\frac{1}{3\beta+1}}}$$

and obtain

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} (s) &= (\beta+1)^{\frac{1}{\beta+1}} (-s)^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_4 \frac{(\bar{\tau} - T)^{\frac{4\beta+1}{\beta+1}}}{R^{4\beta+1}} \bar{x}_{3j}(T)^{-(\beta+1)} A_*^{-\frac{3\beta+1}{\beta+1}} (-s)^{-\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &\quad + \omega_1(R, \delta)(-s) + \omega_2(R, \delta)(-s)^{-\frac{\beta}{\beta+1}} + \omega_3(R, \delta)(-s)^{-\frac{3\beta}{\beta+1}} \end{aligned}$$

where $\omega_p(R, \delta)$, $p = 1, 2, 3$ tends to zero as $R \rightarrow 0$ or $\delta \rightarrow 0$. Furthermore, with the asymptotics

$$\bar{\tau} - T = \frac{R^{\beta+1}}{\beta+1} (1 + o(1)_{R \rightarrow 0}), \quad (4.59)$$

which follow from Lemma 4.6, and the definition of A_* we arrive at

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} (s) &= (\beta+1)^{\frac{1}{\beta+1}} (-s)^{\frac{1}{\beta+1}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B_* (-s)^{\frac{3\beta}{\beta+1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &\quad + \omega_1(R, \delta)(-s) + \omega_2(R, \delta)(-s)^{-\frac{\beta}{\beta+1}} + \omega_3(R, \delta)(-s)^{-\frac{3\beta}{\beta+1}}. \end{aligned}$$

Our goal is to apply Lemma 4.7 with y_p , so we now check its requirements. First, (4.58) and the above considerations are valid for $\bar{\tau} + A_*(R^{4\beta+1}\delta)^{(\beta+1)/(3\beta+1)} s \in [T, \tau_{j-1}^*]$, which by (4.59) and the definition of τ_{j-1}^* means

$$-\frac{1}{A_*} (R^\beta \delta)^{-\frac{\beta+1}{3\beta+1}} \lesssim s \leq -\frac{C}{A_*}.$$

Moreover, we have

$$\frac{d}{ds} y_p = -2y_p^{-\beta} + y_{3-p}^{-\beta} + F_p$$

where

$$F_p = A_*^{\frac{\beta}{\beta+1}} \left(R^{4\beta+1} \delta \right)^{\frac{3\beta}{\beta+1}} x_{3(j+p-2)}.$$

By taking N_* sufficiently large we can make F_p and ω_p small and arrange $s \in [-S, -C/A_*]$ for some S that is large but of order 1. Then, the claim follows from Lemma 4.7, and the case $(-\delta)$ works by interchanging y_1 and y_2 . \square

Corollaries 4.16 and 4.17 allow us to estimate the integrals in Lemma 4.12.

Lemma 4.18 (Estimates for active large particles II). *Let $\beta \geq \beta_*$. In the setting of Proposition 4.15, we have*

$$\int_{T_j}^t \left| x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta} \right| ds \leq C \left(R^{4\beta+1} \delta \right)^{\frac{1}{3\beta+1}}$$

and

$$\int_{T_j}^t \left| x_{\sigma_-(3j)}^{-\beta} - \bar{x}_{\sigma_-(3j)}^{-\beta} \right| ds \leq C \left(R^{4\beta+1} \delta \right)^{\frac{1}{3\beta+1}}$$

for all $t \in [T_j, T_{j-1}]$.

Proof. Since the proofs of both inequalities are identical, we consider only the first. For simplicity of notation we abbreviate

$$\bar{\tau} = \bar{\tau}_{3(j-1)+1}, \quad \tau^* = \tau_{j-1}^* \quad \text{and} \quad R = R_{j-1}.$$

For $t \in [T_j, \tau^*]$ we have $\sigma_+(3(j-1)) = 3(j-1) + 1$, hence Corollary 4.16 and Lemma 4.6 yield

$$\left| x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta} \right| \leq \frac{C|D_1|}{\bar{x}_{3(j-1)+1}^{\beta+1}} \leq \frac{CR^{\beta+1}\delta}{\bar{\tau} - t} \left(\frac{\bar{\tau} - T_j}{\bar{\tau} - t} \right)^{\frac{3\beta}{\beta+1}}.$$

By integration and using Corollary 4.17 we find

$$\begin{aligned} \int_{T_j}^t \left| x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta} \right| ds &\leq CR^{\beta+1}\delta (\bar{\tau} - T_j)^{\frac{3\beta}{\beta+1}} (\bar{\tau} - \tau^*)^{-\frac{3\beta}{\beta+1}} \\ &\leq C \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}}. \end{aligned}$$

For $t \in [\tau^*, T_{j-1}]$, however, we have to deal with the fact that $\sigma_+(3(j-1))$ changes. Here, the lower bound (4.13) implies

$$\int_{\tau^*}^t x_{3(j-1)+p}^{-\beta} ds \leq C \int_{\tau^*}^t (\tau_{3(j-1)+p} - t) \chi_{\{t < \tau_{3(j-1)+p}\}} ds \leq C (\tau_{3(j-1)+p} - \tau^*)^{\frac{1}{\beta+1}}$$

for $p = 1, 2$ and similar inequalities for $\bar{x}_{3(j-1)+p}$. Since the estimate for the possibly large neighbor $\sigma_+(3(j-1)) = 3j$ is even better, the definition of τ^* and Corollary 4.17 yield

$$\int_{\tau^*}^t \left| x_{\sigma_+(3(j-1))}^{-\beta} - \bar{x}_{\sigma_+(3(j-1))}^{-\beta} \right| ds \leq C \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}}$$

for all $t \in [\tau^*, T_{j-1}]$. □

We now combine the estimates from Lemmas 4.11 and 4.12 with the consequences of Proposition 4.15 in order to complete the estimates for all non-vanishing particles.

Corollary 4.19 (Estimates for non-vanishing particles). *Let $\beta \geq \beta_*$. In the setting of Proposition 4.15 we have*

$$\begin{aligned} |D_0(t)| &\leq C \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} + C\delta\delta_*, \\ |D_3(t)| &\leq C \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} + C\delta, \\ \|x - \bar{x}\|_s(t) &\leq C\gamma^{\beta+1} \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} + C\delta\delta_*, \\ \|x - \bar{x}\|_t(t) &\leq CR_{j-1}^{\beta+1}\gamma^{\beta+1} \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} + C\delta\delta_*, \\ \|x - \bar{x}\|_r(t) &\leq CR_{j-1}^{\beta+1} \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} + C\delta \end{aligned}$$

for all $t \in [T_j, T_{j-1}]$.

Proof. Using Lemma 4.18 and the assumptions of Proposition 4.15 in the estimates from Lemma 4.12 we find

$$\begin{aligned} |D_0(t)| &\leq C\delta\delta_* + C(t - T_j) \left(\delta\delta_* + \gamma^{\beta+1}\delta\delta_* \right) + C \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}}, \\ |D_3(t)| &\leq C\delta + C(t - T_j)\delta + C \left(R^{4\beta+1}\delta \right)^{\frac{1}{3\beta+1}} \end{aligned}$$

for all $t \in [T_{j-1}, T_j]$. Then, Lemma 4.11 yields

$$\begin{aligned} \|x - \bar{x}\|_s(t) &\leq C\delta\delta_* + C\frac{t - T_j}{R_{j-2}^{\beta+1}}\delta\delta_* + \frac{C}{R_{j-2}^{\beta+1}} \int_{T_j}^t |D_0(s)| \, ds \\ &\leq C\delta\delta_* + C\frac{t - T_j}{R_{j-2}^{\beta+1}}\delta\delta_* + C\frac{t - T_j}{R_{j-2}^{\beta+1}} \left(R^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}}, \\ \|x - \bar{x}\|_l(t) &\leq C\delta\delta_* + C(t - T_j)\delta\delta_* + C\gamma^{\beta+1}(t - T_j) \left(R^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} \end{aligned}$$

and

$$\|x - \bar{x}\|_r(t) \leq C\delta + C(t - T_j) \left(R^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}}$$

for all $t \in [T_{j-1}, T_j]$. \square

Corollary 4.19 bounds the change of particle differences from above. However, we still need to study how exactly the perturbation is transported from the particles $3j$ to $3(j-1)$ in the time interval $[T_j, T_{j-1}]$. To this end, Lemma 4.20 describes the transfer of mass from $x_{3j}(T_j)$ to $x_{3(j-1)}(T_{j-1})$ and $\bar{x}_{3j}(T_j)$ to $\bar{x}_{3(j-1)}(T_{j-1})$, respectively, while Corollary 4.21 contains the corresponding result for D_0 and D_3 . Of particular interest is the value r_j below, which is the difference between x and \bar{x} and which is responsible for the amplification of the perturbation.

Lemma 4.20. *Denote by*

$$r_j = \begin{cases} +\frac{1}{6}x_{3(j-1)+2}(\tau_{3(j-1)+1}) & \text{if } \tau_{3(j-1)+1} < \tau_{3(j-1)+2}, \\ -\frac{1}{6}x_{3(j-1)+1}(\tau_{3(j-1)+2}) & \text{if } \tau_{3(j-1)+1} > \tau_{3(j-1)+2}, \\ 0 & \text{if } \tau_{3(j-1)+1} = \tau_{3(j-1)+2} \end{cases} \quad (4.60)$$

the scaled size of the remaining particle when the first one of $x_{3(j-1)+p}$, $p = 1, 2$ vanishes. Then we have

$$\begin{aligned} x_{3(j-1)}(T_{j-1}) - x_{3(j-1)}(T_j) &= \frac{2}{3}x_{3(j-1)+1}(T_j) + \frac{1}{3}x_{3(j-1)+2}(T_j) + r_j \\ &\quad + \int_{T_j}^{\min(\tau_{3(j-1)+p})} x_{3(j-2)+2}^{-\beta} - \frac{4}{3}x_{3(j-1)}^{-\beta} + \frac{1}{3}x_{3j}^{-\beta} \, dt \\ &\quad + \int_{\min(\tau_{3(j-1)+p})}^{\max(\tau_{3(j-1)+p})} x_{3(j-2)+2}^{-\beta} - \frac{3}{2}x_{3(j-1)}^{-\beta} + \frac{1}{2}x_{3j}^{-\beta} \, dt \\ &\quad + \int_{\max(\tau_{3(j-1)+p})}^{T_{j-1}} x_{3(j-2)+2}^{-\beta} - 2x_{3(j-1)}^{-\beta} + x_{3j}^{-\beta} \, dt \end{aligned}$$

and

$$\begin{aligned} x_{3j}(T_{j-1}) - x_{3j}(T_j) &= \frac{1}{3}x_{3(j-1)+1}(T_j) + \frac{2}{3}x_{3(j-1)+2}(T_j) - r_j \\ &\quad + \int_{T_j}^{\min(\tau_{3(j-1)+p})} \frac{1}{3}x_{3(j-1)}^{-\beta} - \frac{4}{3}x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} \, dt \\ &\quad + \int_{\min(\tau_{3(j-1)+p})}^{\max(\tau_{3(j-1)+p})} \frac{1}{2}x_{3(j-1)}^{-\beta} - \frac{3}{2}x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} \, dt \\ &\quad + \int_{\max(\tau_{3(j-1)+p})}^{T_{j-1}} x_{3(j-1)}^{-\beta} - 2x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} \, dt, \end{aligned}$$

where the maximum and minimum in the integral boundaries are taken over $p = 1, 2$ and integrals with identical lower and upper boundary are 0.

Proof. The assertions follow from direct computations considering the two cases $\tau_{3(j-1)+1} \leq \tau_{3(j-1)+2}$ and $\tau_{3(j-1)+1} > \tau_{3(j-1)+2}$ separately. Since the arguments in both cases are the same we only go through the first, where $T_j < \tau_{3(j-1)+1} \leq \tau_{3(j-1)+2} \leq T_{j-1}$.

For $t \in (T_j, \tau_{3(j-1)+1})$ we have $x_k(t) > 0$ for $k \leq 3j$ and $k = 3(j+1)$, while $x_{3j+1}(t) = x_{3j+2}(t) = 0$. Hence, the equations for x_k , $k = 3(j-1)$, $3(j-1)+1$, $3(j-1)+2$, $3j$ read

$$\dot{x}_{3(j-1)} = x_{3(j-2)+2}^{-\beta} - 2x_{3(j-1)}^{-\beta} + x_{3(j-1)+1}^{-\beta}, \quad (4.61)$$

$$\dot{x}_{3(j-1)+1} = x_{3(j-1)}^{-\beta} - 2x_{3(j-1)+1}^{-\beta} + x_{3(j-1)+2}^{-\beta}, \quad (4.62)$$

$$\dot{x}_{3(j-1)+2} = x_{3(j-1)+1}^{-\beta} - 2x_{3(j-1)+2}^{-\beta} + x_{3j}^{-\beta}, \quad (4.63)$$

$$\dot{x}_{3j} = x_{3(j-1)+2}^{-\beta} - 2x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta}. \quad (4.64)$$

Adding twice (4.62) to (4.63) gives

$$2\dot{x}_{3(j-1)+1} + \dot{x}_{3(j-1)+2} = 2x_{3(j-1)}^{-\beta} - 3x_{3(j-1)+1}^{-\beta} + x_{3j}^{-\beta}, \quad (4.65)$$

and using (4.65) to eliminate $x_{3(j-1)+1}^{-\beta}$ in (4.61) we obtain

$$\dot{x}_{3(j-1)} = -\frac{2}{3}\dot{x}_{3(j-1)+1} - \frac{1}{3}\dot{x}_{3(j-1)+2} + x_{3(j-2)+2}^{-\beta} - \frac{4}{3}x_{3(j-1)}^{-\beta} + \frac{1}{3}x_{3j}^{-\beta}.$$

Integration from T_j to $\tau_{3(j-1)+1}$ and the vanishing of $x_{3(j-1)+1}(\tau_{3(j-1)+1})$ then yield

$$\begin{aligned} x_{3(j-1)}(\tau_{3(j-1)+1}) - x_{3(j-1)}(T_j) &= \frac{2}{3}x_{3(j-1)+1}(T_j) + \frac{1}{3}x_{3(j-1)+2}(T_j) \\ &\quad - \frac{1}{3}x_{3(j-1)+2}(\tau_{3(j-1)+1}) + \int_{T_j}^{\tau_{3(j-1)+1}} x_{3(j-2)+2}^{-\beta} - \frac{4}{3}x_{3(j-1)}^{-\beta} + \frac{1}{3}x_{3j}^{-\beta} dt. \end{aligned} \quad (4.66)$$

Similarly, adding twice (4.63) to (4.62), using the result to eliminate $x_{3(j-1)+2}^{-\beta}$ in (4.64), and integrating, we find

$$\begin{aligned} x_{3j}(\tau_{3(j-1)+1}) - x_{3j}(T_j) &= \frac{1}{3}x_{3(j-1)+1}(T_j) + \frac{2}{3}x_{3(j-1)+2}(T_j) \\ &\quad - \frac{2}{3}x_{3(j-1)+2}(\tau_{3(j-1)+1}) + \int_{T_j}^{\tau_{3(j-1)+1}} \frac{1}{3}x_{3(j-1)}^{-\beta} - \frac{4}{3}x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} dt. \end{aligned} \quad (4.67)$$

Second, for $\tau_{3(j-1)+1} < \tau_{3(j-1)+2}$ and $t \in (\tau_{3(j-1)+1}, \tau_{3(j-1)+2})$ the non-vanished particles $k = 3(j-1)$, $3(j-1)+2$, $3j$ satisfy

$$\dot{x}_{3(j-1)} = x_{3(j-2)+2}^{-\beta} - 2x_{3(j-1)}^{-\beta} + x_{3(j-1)+2}^{-\beta}, \quad (4.68)$$

$$\dot{x}_{3(j-1)+2} = x_{3(j-1)}^{-\beta} - 2x_{3(j-1)+2}^{-\beta} + x_{3j}^{-\beta}, \quad (4.69)$$

and (4.64). Again, using (4.69) to eliminate $x_{3(j-1)+2}^{-\beta}$ in (4.68) and (4.64) and integrating, we obtain

$$\begin{aligned} x_{3(j-1)}(\tau_{3(j-1)+2}) - x_{3(j-1)}(\tau_{3(j-1)+1}) &= \\ \frac{1}{2}x_{3(j-1)+2}(\tau_{3(j-1)+1}) + \int_{\tau_{3(j-1)+1}}^{\tau_{3(j-1)+2}} x_{3(j-2)+2}^{-\beta} - \frac{3}{2}x_{3(j-1)}^{-\beta} + \frac{1}{2}x_{3j}^{-\beta} dt \end{aligned} \quad (4.70)$$

and

$$\begin{aligned} x_{3j}(\tau_{3(j-1)+2}) - x_{3j}(\tau_{3(j-1)+1}) &= \\ \frac{1}{2}x_{3(j-1)+2}(\tau_{3(j-1)+1}) + \int_{\tau_{3(j-1)+1}}^{\tau_{3(j-1)+2}} \frac{1}{2}x_{3(j-1)}^{-\beta} - \frac{3}{2}x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} dt. \end{aligned} \quad (4.71)$$

Finally, integrating the equations for $x_{3(j-1)}$ and x_{3j} over $t \in (\tau_{3(j-1)+2}, T_{j-1})$ yields

$$x_{3(j-1)}(T_{j-1}) - x_{3(j-1)}(\tau_{3(j-1)+2}) = \int_{\tau_{3(j-1)+2}}^{T_{j-1}} x_{3(j-2)+2}^{-\beta} - 2x_{3(j-1)}^{-\beta} + x_{3j}^{-\beta} dt, \quad (4.72)$$

$$x_{3j}(T_{j-1}) - x_{3j}(\tau_{3(j-1)+2}) = \int_{\tau_{3(j-1)+2}}^{T_{j-1}} x_{3(j-1)}^{-\beta} - 2x_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} dt, \quad (4.73)$$

and adding (4.66), (4.70), (4.72) as well as (4.67), (4.71), (4.73) proves the claim. \square

Lemma 4.20 holds as well for \bar{x} with $r_j = 0$ identically. Taking the difference of x and \bar{x} we obtain the following result.

Corollary 4.21. *Suppose that either $\tau_{3(j-1)+1} < \bar{\tau}_{3(j-1)+1} < \tau_{3(j-1)+2}$ or $\tau_{3(j-1)+2} < \bar{\tau}_{3(j-1)+1} < \tau_{3(j-1)+1}$. Then we have*

$$D_0(T_{j-1}) = D_0(T_j) + \frac{2}{3}D_1(T_j) + \frac{1}{3}D_2(T_j) + r_j + \mathcal{I}_j$$

and

$$D_3(T_{j-1}) = D_3(T_j) + \frac{1}{3}D_1(T_j) + \frac{2}{3}D_2(T_j) - r_j + \mathcal{J}_j,$$

where r_j is as in (4.60) and

$$\begin{aligned} \mathcal{I}_j &= \mathcal{I}_j^1 + \mathcal{I}_j^2 + \mathcal{I}_j^3 \\ &= \int_{T_j}^{\min(\tau_{3(j-1)+p})} \left(x_{3(j-2)+2}^{-\beta} - \bar{x}_{3(j-2)+2}^{-\beta} \right) - \frac{4}{3}D_0^\beta + \frac{1}{3}D_3^\beta dt \\ &\quad + \int_{\min(\tau_{3(j-1)+p})}^{\bar{\tau}_{3(j-1)+1}} x_{3(j-2)+2}^{-\beta} - \bar{x}_{3(j-2)+2}^{-\beta} - \frac{3}{2}x_{3(j-1)}^{-\beta} + \frac{4}{3}\bar{x}_{3(j-1)}^{-\beta} + \frac{1}{2}x_{3j}^{-\beta} - \frac{1}{3}\bar{x}_{3j}^{-\beta} dt \\ &\quad + \int_{\bar{\tau}_{3(j-1)+1}}^{T_{j-1}} x_{3(j-2)+2}^{-\beta} - \bar{x}_{3(j-2)+2}^{-\beta} - \frac{3}{2}x_{3(j-1)}^{-\beta} + 2\bar{x}_{3(j-1)}^{-\beta} + \frac{1}{2}x_{3j}^{-\beta} - \bar{x}_{3j}^{-\beta} dt \end{aligned}$$

as well as

$$\begin{aligned} \mathcal{J}_j &= \mathcal{J}_j^1 + \mathcal{J}_j^2 + \mathcal{J}_j^3 \\ &= \int_{T_j}^{\min(\tau_{3(j-1)+p})} \frac{1}{3}D_0^\beta - \frac{4}{3}D_3^\beta + \left(x_{3(j+1)}^{-\beta} - \bar{x}_{3(j+1)}^{-\beta} \right) dt \\ &\quad + \int_{\min(\tau_{3(j-1)+p})}^{\bar{\tau}_{3(j-1)+1}} \frac{1}{2}x_{3(j-1)}^{-\beta} - \frac{1}{3}\bar{x}_{3(j-1)}^{-\beta} - \frac{3}{2}x_{3j}^{-\beta} + \frac{4}{3}\bar{x}_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} - \bar{x}_{3(j+1)}^{-\beta} dt \\ &\quad + \int_{\bar{\tau}_{3(j-1)+1}}^{T_{j-1}} \frac{1}{2}x_{3(j-1)}^{-\beta} - \bar{x}_{3(j-1)}^{-\beta} - \frac{3}{2}x_{3j}^{-\beta} + 2\bar{x}_{3j}^{-\beta} + x_{3(j+1)}^{-\beta} - \bar{x}_{3(j+1)}^{-\beta} dt. \end{aligned}$$

Using our previous estimates for the various terms in Corollary 4.21 we can finally characterize $D_0(T_{j-1})$ and $D_3(T_{j-1})$.

Proposition 4.22. *Suppose that $\beta \geq \beta_*$. There is a constant $a_* > 0$ such that in the setting of Proposition 4.15 we have*

$$|r_j| = a_* \left(R_{j-1}^{4\beta+1} \delta \right)^{\frac{1}{3\beta+1}} (1 + o(1)_{R_{j-1} \rightarrow 0}).$$

Moreover, we have

$$|D_0(T_{j-1}) - r_j| \leq \left(R_{j-1}^{4\beta+1} \delta \right)^{\frac{1}{3\beta+1}} o(1)_{\gamma \rightarrow 0, R_{j-1} \rightarrow 0} + O \left(R_{j-1}^{\beta+1} \delta (1 + \delta_*) \right) + \delta \delta_* (1 + R_{j-1}^{\beta+1})$$

and

$$|D_3(T_{j-1}) + r_j| \leq \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} o(1)_{R_{j-1} \rightarrow 0} + O\left(R_{j-1}^{\beta+1}\delta(1+\delta_*)\right) + \delta + R_{j-1}^{\beta+1}\delta\delta_*.$$

Proof. First, to determine the size of r_j note that after one of the small particles $x_{3(j-1)+p}$, $p = 1, 2$ has vanished, the remaining one has large neighbors only. Therefore, the power law from Lemma 4.6 (formally doubling the small particle) and Corollary 4.17 imply

$$\begin{aligned} |r_j| &= \frac{1}{6}(\beta+1)^{\frac{1}{\beta+1}} |\tau_{3(j-1)+2} - \tau_{3(j-1)+1}|^{\frac{1}{\beta+1}} \left(1 + O(R_{j-1}^{\beta+1})\right) \\ &= \frac{1}{3}(\beta+1)^{\frac{1}{\beta+1}} A_* \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} (1 + o(1)_{R_{j-1} \rightarrow 0}). \end{aligned}$$

As A_* depends only on β and $\bar{x}_{3j}(T_j)$, and as the latter satisfies $\bar{x}_{3j}(T_j) = 1 + O(R_{j-1})$ by the Constancy Lemma 4.10 we obtain

$$|r_j| = a_* \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} (1 + o(1)_{R_{j-1} \rightarrow 0})$$

for a constant a_* .

Next, we estimate the other contributions to D_0 and D_3 from Corollary 4.21 separately, and since the proofs for both differences are similar, we consider only D_0 . First, by assumptions (4.37) we clearly have

$$|D_0(T_j) + \frac{2}{3}D_1(T_j) + \frac{1}{3}D_2(T_j)| \leq \delta\delta_* + R_{j-1}^{\beta+1}\delta\delta_*.$$

Furthermore, linearizing in \mathcal{I}_j^1 we have

$$|\mathcal{I}_j^1| \leq C \int_{T_j}^{\min(\tau_{3(j-1)+p})} \frac{1}{R_{j-1}^{\beta+1}} \|x - \bar{x}\|_s + |D_0| + |D_3| \, dt$$

and using Corollary 4.19 we find

$$\begin{aligned} |\mathcal{I}_j^1| &\leq C \int_{T_j}^{\min(\tau_{3(j-1)+p})} \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} \left(1 + \frac{\gamma^{\beta+1}}{R_{j-2}^{\beta+1}}\right) + C\delta(1+\delta_*) \, dt \\ &\leq C \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} \left(\gamma^{2(\beta+1)} + R_{j-1}^{\beta+1}\right) + CR_{j-1}^{\beta+1}\delta(1+\delta_*). \end{aligned}$$

Finally, \mathcal{I}_j^2 and \mathcal{I}_j^3 are easily estimated using the Constancy Lemma 4.10 for $x_{3(j-2)+2}$ and the fact that all other particles that appear in the integrands remain large. The upshot is that

$$\begin{aligned} |\mathcal{I}_j^2| + |\mathcal{I}_j^3| &\leq \left(CR_{j-2}^{-\beta} + 1\right) (|\tau_{3(j-1)+1} - \bar{\tau}_{3(j-1)+1}| + |\tau_{3(j-1)+2} - \bar{\tau}_{3(j-1)+1}|) \\ &\leq CR_{j-1}^{-\beta} \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{\beta+1}{3\beta+1}} \\ &= C \left(R_{j-1}^{4\beta+1}\delta\right)^{\frac{1}{3\beta+1}} \left(R_{j-1}^{\beta}\delta\right)^{\frac{\beta}{3\beta+1}}, \end{aligned}$$

where we have used Corollary 4.17 in the second inequality. \square

Theorem 4.9 is now a consequence of Corollary 4.19 and Proposition 4.22, once we choose γ_* sufficiently small and N_* , C_* sufficiently large.

4.5 Proof of Theorem 4.1

We now look for an appropriate sequence (ε^N) and send $N \rightarrow \infty$. Clearly, by restricting ourselves to a not relabeled subsequence we may assume that $R_{j,p}^N$ converges to some $R_{j,p} \geq R_j/2$ as $N \rightarrow \infty$ for all $j \in \mathbb{N}$ and $p = 1, 2$.

Suppose for the moment that we already have some ε^N for large N and that we can apply Theorem 4.9 iteratively to obtain a sequence $\delta_{j-1} = \bar{\delta}(\delta_j)$ for $j = N_* + 1, \dots, N$ and $\delta_N = \varepsilon^N$. Then we have

$$\frac{3}{4}a_* \left(R_{j-1}^{4\beta+1} \delta_j \right)^{\frac{1}{3\beta+1}} \leq \delta_{j-1} \leq \frac{5}{4}a_* \left(R_{j-1}^{4\beta+1} \delta_j \right)^{\frac{1}{3\beta+1}},$$

and after taking logarithms these inequalities become

$$\frac{\ln \delta_j}{3\beta+1} + C_1 + C_3 \ln R_{j-1} \leq \ln \delta_{j-1} \leq \frac{\ln \delta_j}{3\beta+1} + C_2 + C_3 \ln R_{j-1}, \quad (4.74)$$

where $C_1 = \ln(3a_*/4)$, $C_2 = \ln(5a_*/4)$ and $C_3 = (4\beta+1)/(3\beta+1)$. Iterating (4.74) we arrive at

$$\frac{\ln \varepsilon^N}{(3\beta+1)^{N+1-j}} + S_1(N, j) \leq \ln \delta_{j-1} \leq \frac{\ln \varepsilon^N}{(3\beta+1)^{N+1-j}} + S_2(N, j) \quad (4.75)$$

where

$$S_p(N, j) = \sum_{k=0}^{N-j} \frac{C_p + C_3 \ln R_{j+k-1}}{(3\beta+1)^k} = \sum_{k=0}^{N-j} \frac{C_p + C_3(j+k-1) \ln \gamma}{(3\beta+1)^k}$$

due to $R_j = \gamma^j$. For fixed j the sums $S_p(N, j)$ converge absolutely as $N \rightarrow \infty$ and we have

$$|S_p(N, j)| \leq C + C |\ln \gamma| (j+1) \quad (4.76)$$

for all $N \geq j$ with constants independent of N . In particular, for $j = N_* + 1$ we have

$$\frac{\ln \varepsilon^N}{(3\beta+1)^{N-N_*}} + S_1(N, N_* + 1) \leq \ln \delta_{N_*} \leq \frac{\ln \varepsilon^N}{(3\beta+1)^{N-N_*}} + S_2(N, N_* + 1),$$

which shows that for fixed $\delta_{N_*} > 0$ we can find $\varepsilon^N > 0$ for $N > N_*$ so that

$$\ln \varepsilon^N \sim (3\beta+1)^{N-N_*} \ln \delta_{N_*} \quad (4.77)$$

and so that the iterative procedure $\delta_{j-1} = \bar{\delta}(\delta_j)$ for $j = N_* + 1, \dots, N$ yields the prescribed value δ_{N_*} , provided that the assumptions of Theorem 4.9 are satisfied. Moreover, from (4.75) we know that

$$\begin{aligned} & -3\beta \frac{\ln \varepsilon^N}{(3\beta+1)^{N+1-j}} + S_1(N, j) - S_2(N, j+1) \\ & \leq \ln \frac{\delta_{j-1}}{\delta_j} \leq -3\beta \frac{\ln \varepsilon^N}{(3\beta+1)^{N+1-j}} - S_1(N, j+1) + S_2(N, j), \end{aligned}$$

and using (4.77)–(4.76) we infer that

$$(3\beta+1)^{j-1+N_*} |\ln \delta_{N_*}| - C(j+1) \leq \ln \frac{\delta_{j-1}}{\delta_j} \leq (3\beta+1)^{j-1+N_*} |\ln \delta_{N_*}| + C(j+1) \quad (4.78)$$

for $j = N_* + 1, \dots, N-1$. Since the first term on the left and on the right of (4.78) grows faster in j than the second we can choose δ_{N_*} so small that $|\ln \delta_{N_*}| \gg C(N_* + 1)$ and

$$\ln \frac{\delta_{j-1}}{\delta_j} \geq \ln C_* \quad \Longleftrightarrow \quad \delta_{j-1} \geq C_* \delta_j$$

for $j = N_* + 1, \dots, N - 1$, where C_* is from Theorem 4.9. We also obtain $C_*\delta_N \leq \delta_{N-1}$ by using (4.77) in (4.74). Hence, for sufficiently small δ_{N_*} and ε^N chosen appropriately as in (4.77), Theorem 4.9 applies for all $j = N_* + 1, \dots, N$ and all $N > N_* + 1$. The iteration over j yields

$$|x_{3N_*}^N(T_{N_*}^N) - \bar{x}_{3N_*}^N(T_{N_*}^N)| = \delta_{N_*} > 0 \quad (4.79)$$

for all $N > N_* + 1$.

Using our compactness results from Section 3 as well as $\varepsilon^N \rightarrow 0$ as $N \rightarrow \infty$, we pass to the limit along a subsequence of N and find two solutions x and \bar{x} of (2.2) with the same initial data. Moreover, since $cR_{N_*} \leq T_{N_*}^N \leq CR_{N_*}$, we also have $T_{N_*}^N \rightarrow T_{N_*} > 0$ as $N \rightarrow \infty$, and (4.79) yields

$$|x_{3N_*}(T_{N_*}) - \bar{x}_{3N_*}(T_{N_*})| = \delta_{N_*} > 0.$$

Then by continuity of x_{3N_*} and \bar{x}_{3N_*} both differ in a time interval around T_{N_*} .

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